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PROFESSOR

ANALYTIC AND PRACTICAL
PLANE AND SPHERICAL
TRIGONOMETRY

For the use of Colleges and Schools.

By

REV. L. P. PAQUIN, O. M. I.

PROFESSOR OF CIVIL ENGINEERING IN THE COLLEGE OF OTTAWA.

OTTAWA.

1878
Printed by the
Sisters of the GOOD SHEPHERD.

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PREFACE.

The following treatise, designed for colleges and high schools, and more especially for students in Engineering, is prepared upon a thoroughly new method— It embodies everything necessary for a course of Trigonometrical study sufficiently extensive to enable the student to comprehend readily any applications of Trigonometry he may meet with in the best modern works on speculative and applied mathematics.

A particular attention has been paid to the analytical department of the subject. Chapter IV, especially devoted to this part, embraces the most important trigonometrical formulæ which are found in extensive treatises, together with several other useful formulæ not usually found in works devoted to analytical Trigonometry. The Author keeping in view the logical and progressive development of the subject, has endeavoured to arrange a complete series of analytical deductions according to a very methodical order, which he believes to be a new one.

Those who wish to limit their studies to the relations existing between the several parts of a triangle, in as much as it is required for the practice of ordinary Land Surveying, may leave out Chapter IV and Chapter V without breaks in continuity in so far as the object they have in view may be concerned.

The investigation of Trigonometric series being more easily carried on with the aid of Differential Calculus than with the ordinary methods, the Author has given such of its principles, in the beginning of Chapter IX, as are necessary for the intelligence of his investigations on this part of the subject.

THE AUTHOR.

College of Ottawa, January 1878.

ERRATA.

Page	16	line 20 and 21; for + read ×
"	28	" 18 for positive read negative
"	"	" 21 for negative read positive.
"	55	" 4 for $\sin 2A$ read $\frac{1}{\sin 2A}$
"	64	" 19 for $\cos (A + A)$ read $\cos (A + B)$
"	66	" 1 for $\operatorname{cosec} B + \operatorname{cosec} A$ read $\operatorname{cosec} B - \operatorname{cosec} A$
"	"	" 12 for $2 \sin A \sin B$ read $-2 \sin A \sin B$
"	"	" 16 for $\tan B$ read $\tan^2 B$
"	72	" 10 for $\sec^2 A \sec^2 B + \operatorname{cosec}^2 A \operatorname{cosec}^2 B$ read $\sec^2 A \sec^2 B$ $\operatorname{cosec}^2 A \operatorname{cosec}^2 B$
"	74	" 16 for $\tan (A + B)$ read $\sin (A + B)$
"	81	" 28 before the third sine read +
"	87	" 5 for $\sin - B$ read $\sin 2B$
"	111	" 5 for $\sin \frac{1}{2}(A + B)$ read $\sin \frac{1}{2}(A + C)$
"	112	" 3 for (555) read (255)
"	"	" 13 for (459) read (500)
"	"	the second members of (507) and (508) must be interchanged,
"	113	the second members of [511] and [512] must be interchanged.

ANGULAR

MUTUAL

VARIATION

ANGULAR

INVERSE

RELATIONS
TRIANGLE

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PART I.

PLANE TRIGONOMETRY.

OBJECT OF TRIGONOMETRY.

TRIGONOMETRY, in its original and restricted sense, teaches how to investigate the relations subsisting between the angles and the sides of triangles.

Plane Trigonometry treats of plane triangles; Spherical Trigonometry treats of spherical triangles.

In modern analysis, Trigonometry has a much wider meaning: its object having been extended to all algebraic investigations relating to plane angles whether they belong to triangles or not, and its formulæ, being extensively employed as instruments of calculation in almost every department of scientific investigation.

Plane Trigonometry is called *Practical* when restricted to the computation of triangles, and *Analytical* when taken in its enlarged signification,

ANALITICAL TRIGONOMETRY may be defined *that branch of Mathematics whose object is to investigate the nature and properties of angular functions, and the relations existing between them.*

CHAPTER I.

A *Function* implies such a connection between two quantities that one of them undergoes a change of value when the other, upon which the former is dependent, passes from one state of magnitude to another.

Angular Functions are magnitudes which depend upon the angular magnitude, that is: upon the variable angle that may be conceived to extend in the whole angular space.

Angular Space is the space which may be run through by a straight line revolving entirely about a fixed point in a plane. The whole angular space is divided into 360 equal parts called *degrees*: each degree is divided into 60 equal parts called *minutes*: each minute is divided into 60 equal parts called *seconds*. Degrees, minutes and seconds are designated by the following marks: ° ' " thus 25° 40' 45" is read 25 degrees, 40 minutes, 45 seconds. And lower subdivisions of a degree are usually expressed as decimals of a second.

Two straight lines drawn at right angles, divide the whole angular space into four equal parts called *quadrants*. It is obvious that a quadrant contains 90°; therefore, a right angle = 90°.

If with a radius of any length, generally taken as unity, a circle be described about the point of intersection of these two lines as centre, the circumference of this circle will be the linear representative of the whole angular space about that point. And since the angles at the centre of a circle are proportional to the arcs of the circumference intercepted between their sides, and are measured by them, the angular functions may, also, be conceived as dependent upon these arcs: under



In this respect, they are called *circular functions* and, also, *trigonometrical lines*, which will be defined hereafter.

The complement of an angle or arc is the difference between 90° and that angle or arc.

The supplement of an angle or arc is the difference between 180° and that angle or arc.

We may take the following method of illustrating the generation of the angular magnitude,

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Let angles be conceived as generated by the motion of a straight line revolving from a fixed line CA , about a fixed point C : when the revolving line coincides with CA , there is no angle formed, or the angle = 0



When it assumes successively the positions of $CP, CB, CP', CA, CP'', CB', CP'''$, the angle generated, are respectively an acute angle, a right angle, an obtuse angle less than two right angles, an angle equal to two right angles, an angle greater than two and less than three right angles, an angle equal to three right angles, an angle greater than three and less than four right angles.

When, finally, the generating line has completed an entire revolution, it will coincide again with CA , having formed an angle equal to four right angles.

If we suppose the revolution to recommence, then the generating line will form angles greater than four, than five, or any other number of right angles.

Now any point A on the moving line, will, at the same time, describe the circumference $ABA'B'$ about C , and may be considered as the generator of all arcs that may be taken on the circumference.

Let A be the origin of any arc, and the arcs to be generated by the motion of a point in the direction upwards, then AA' will be called the diameter passing through the origin, B the origin of the complements, since the complement of an arc is the remainder obtained by subtracting that arc from a quadrant or 90° , BB' the diameter passing through that origin; — A' the origin of the supplements, and AA' the diameter passing through the origin of the supplements.

Where the generating point is conceived to stop in its motion, there is the end of the arc generated. Thus the generating point being at P , the arc generated is AP , its complement BP , its supplement $A'P$.

The generating point being at P' , the arc generated is AP' , its complement BP' , its supplement $A'P'$.

The generating point being at P'' , the arc generated is AP'' , its complement BP'' , its supplement $A'P''$.

The generating point being at P''' the arc generated is AP''' its complement BP''' , its supplement $A'P'''$.

Hence it is seen that the points P, P', P'', P''' , are respectively the ends of the arcs AP, AP', AP'', AP''' , and of their complements and their supplements; that is *the end of an arc, that of its complement and of its supplement are the same point.*

It is obvious also that the complement of an arc less than 90° is positive, and that the complement of an arc greater than 90° is negative, because by the rule of algebraic subtraction, we have

$$90 - [90 - a] = b$$

$$90 - [90 + a] = -b$$

For the same reason the supplement of an arc less than 180° is positive and that of an arc greater than 180° is negative, that is:

$$180 - [180 - a] = b$$

$$180 - [180 + a] = -b$$

The **SINE** of an arc is the perpendicular let fall from the end of the arc to the diameter passing through its origin.

The **TANGENT** of an arc is the perpendicular erected at the origin of the arc on the diameter passing through the origin and terminated by its intersection with a line drawn from the centre through the end of the arc.

The **SECANT** of an arc is the line drawn from the centre through the end of the arc, and terminated by its intersection with the tangent.

The **VERSED SINE** of an arc is that part of the diameter passing through the origin, intercepted between the origin and the foot of the sine.

Thus for the sake of illustration we have in the annexed figure:

$PM = \text{sine of the arc } AP$

$AE = \text{tangent} \quad " \quad "$

$CE = \text{secant} \quad " \quad "$

$AM = \text{versed sine} \quad " \quad "$

The same definitions being applied to the trigonometric lines of



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the arc BP , complement of AP , and whose origin is at B , we have

$PL = \text{sine of the arc } BP$

$BR = \text{tangent " "}$

$CR = \text{secant " "}$

$BL = \text{versed sine " "}$

But because the arc BP is the complement of AP , we have:

$PL = \text{sine of the complement of } AP$

$BR = \text{tangent " "}$

$CR = \text{secant " "}$

$BL = \text{versed sine " "}$

And for abbreviation we write, by a contraction of the word complement:

cosine for sine of complement

cotangent for tangent of complement

cosecant for secant of complement

covered sine for versed sine of complement

And for further abbreviation, we write

sin for sine

cos for cosine

tan for tangent

cot for cotangent

sec for secant

cosec for cosecant

covers for covered sine

Because the sine of an arc is always parallel to the diameter BB' passing through the origin of the complements, it follows that the cosine PL is equal to that part of the diameter AA' intercepted between the centre and the foot of the sine; and therefore in practice the cosine is taken as being CM , and is defined the *distance between the centre and the foot of the sine*.

The trigonometric lines of an arc, or CIRCULAR FUNCTIONS, depend on the magnitude of the radius of the circle; for the greater is the radius of a circle the greater will be the arc included between the sides of an angle at the centre, and the greater also will be the circular functions relating to that arc, as is evident by the inspection of the annexed figure,

But from the similitude of triangles, we have

$$\begin{aligned}\frac{pm}{Cp} &= \frac{PM}{CP} \\ \frac{ae}{Ca} &= \frac{AE}{CA} \\ \text{etc.}\end{aligned}$$

that is, the ratios of circular functions of an arc of a greater circle to the radius of that circle are equal to the ratios of circular functions of an arc of a smaller circle to its radius.

These ratios constitute properly the **ANGULAR FUNCTIONS** of the angles subtended by the arcs, so that the sine of an angle is the ratio of the sine of the included arc to the radius of the circle; and in general *the angular functions are the ratios of the corresponding circular functions to the radius of the circle.*

It is obvious, from the last definition, that the angular functions are independent of the magnitude of the radius.

But since in practice, the radius of the circle is always assumed as unity, then the angular functions are equal to the corresponding circular functions; and it must be borne in mind that "circular functions" "angular functions" and "trigonometric lines" all designate the same magnitudes.



Since
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Therefore

CHAPTER II

MUTUAL RELATIONS BETWEEN ANGULAR FUNCTIONS

Formulas for Sine and Cosine



N. B. In the following investigations, the radius will be assumed as unity.

$$\begin{aligned} \text{Since } PM^2 + CM^2 &= CP^2 \\ \text{or } \sin^2 + \cos^2 &= 1 \quad [a] \\ \text{Therefore } \sin &= \sqrt{1 - \cos^2} \end{aligned}$$

$$\begin{aligned} \text{Since } CP : PM :: CE : AE \\ \text{or } 1 : \sin :: \sec : \tan \quad [b] \\ \text{Therefore } \sin &= \frac{\tan}{\sec} \quad [2] \end{aligned}$$

$$\begin{aligned} \text{Since } CB : PM :: BR : CM \\ \text{or } 1 : \sin :: \cot : \cos \quad [c] \\ \text{Therefore } \sin &= \frac{\cos}{\cot} \quad [3] \end{aligned}$$

$$\begin{aligned} \text{Since } CA : CM :: AE : PM \\ \text{or } 1 : \cos :: \tan : \sin \quad [d] \\ \text{Therefore } \sin &= \tan \times \cos \quad [4] \end{aligned}$$

$$\begin{aligned} \text{Since } CB : PM :: CR : CP \\ \text{or } 1 : \sin :: \operatorname{cosec} : 1 \quad [e] \\ \text{Therefore } \sin &= \frac{1}{\operatorname{cosec}} \quad [5] \end{aligned}$$

By the definition of the trigonometrical lines, we know that the sine, tangent and secant of an angle or arc, are equal to the cosine, cotangent and cosecant of the complement of that angle or arc. Thus denoting by a any assumed arc, we have

$$\sin (90^\circ - a) = \cos a$$

$$\tan (90^\circ - a) = \cot a$$

$$\sec (90^\circ - a) = \operatorname{cosec} a$$

These relations give rise to the following deductions:

Because $\sin = \sqrt{[1 - \cos^2]}$, by (1)
we have $\sin (90^\circ - a) = \sqrt{[1 - \cos^2 (90^\circ - a)]}$

hence $\cos a = \sqrt{[1 - \sin^2 a]}$

or generally, $\cos = \sqrt{[1 - \sin^2]}$

Deduction verified by the equation $[a]$

(6)

Because $\sin = \frac{\tan}{\sec}$, by (2)

we have $\sin (90^\circ - a) = \frac{\tan (90^\circ - a)}{\sec (90^\circ - a)}$

hence $\cos a = \frac{\cot a}{\operatorname{cosec} a}$

or generally, $\cos = \frac{\cot}{\operatorname{cosec}}$

(7)

Deduction verified by the proportion

$$CP : PL = CM :: CR : BR$$

or $1 : \cos :: \operatorname{cosec} : \cot [f]$

Because $\sin = \frac{\cos}{\cot}$, by [3]

we have $\sin (90^\circ - a) = \frac{\cos (90^\circ - a)}{\cot (90^\circ - a)}$

hence $\cos a = \frac{\sin a}{\tan a}$

or generally $\cos = \frac{\sin}{\tan}$

(8)

Deduction verified by the proportion $[d]$

know that the
to the cosine,
le or arc. Thus

Because $\sin = \tan \times \cos$, by (4)
we have $\sin (90^\circ - a) = \tan (90^\circ - a) \times \cos (90^\circ - a)$
hence $\cos a = \cot a \times \sin a$
or generally, $\cos = \cot \times \sin$ (9)
Deduction verified by the proportion [c]

Because $\sin = \frac{1}{\operatorname{cosec}}$ —by [5]
we have $\sin (90^\circ - a) = \frac{1}{\operatorname{cosec} (90^\circ - a)}$
(6) hence $\cos a = \frac{1}{\sec a}$
or generally, $\cos = \frac{1}{\sec}$ (10)
Deduction verified by the proportion
 $CA : CM :: CE : CP$
or $1 : \cos :: \sec : 1$

Formulas for tangent and cotangent

(7) By the Proportion [d] we have
 $\tan = \frac{\sin}{\cos}$ (11)

Since $CA : AE :: BR : CB$
or $1 : \tan :: \cot : 1$ [g]
we have $\tan = \frac{1}{\cot}$ (12)

By the proportion [b] we have
 $\tan = \sin \times \sec$ (13)

Since $CE^2 = AE^2 + CA^2$
or $\sec^2 = \tan^2 + 1$ [h]
(8) we have $\tan = \sqrt{\sec^2 - 1}$ (14)

Because $\tan a = \frac{\sin a}{\cos a}$ by (11)

we have $\tan (90^\circ - a) = \frac{\sin (90^\circ - a)}{\cos (90^\circ - a)}$

hence $\cot a = \frac{\cos a}{\sin a}$, see page 14

or generally, $\cot = \frac{\cos}{\sin}$ (15)

Deduction verified by the proportion [c]

Because $\tan = \frac{1}{\cot}$

we have $\tan (90^\circ - a) = \frac{1}{\cot (90^\circ - a)}$

hence $\cot a = \frac{1}{\tan a}$

or generally, $\cot = \frac{1}{\tan}$ (16)

Deduction verified by the proportion [g]

Because $\tan = \sin \times \sec$, by (13)

we have $\tan (90^\circ - a) = \sin (90^\circ - a) \times \sec (90^\circ - a)$

hence $\cot = \cos a + \operatorname{cosec} a$

or generally, $\cot = \cos + \operatorname{cosec}$ (17)

Deduction verified by the proportion [f]

Because $\tan = \sqrt{[\sec^2 - 1]}$

we have $\tan (90^\circ - a) = \sqrt{[\sec^2 (90^\circ - a) - 1]}$

hence $\cot a = \sqrt{[\operatorname{cosec}^2 a - 1]}$

or generally, $\cot = \sqrt{[\operatorname{cosec}^2 - 1]}$ (18)

Deduction verified by the equation

$$CR^2 = BR^2 + CB^2$$

or $\operatorname{cosec}^2 = \cot^2 + 1$ [i]

Formulas for Secant and cosecant.

By the equation [h] we have

$$\sec = \sqrt{1 + \tan^2} \quad (19)$$

By the proportion 1 : cos :: sec : 1, see page 15, we have

$$\sec = \frac{1}{\cos} \quad (20)$$

By the proportion [b] we have

$$\sec = \frac{\tan}{\sin} \quad (21)$$

Because

$$\sec = \sqrt{1 + \tan^2}, \text{ by (19)}$$

we have

$$\sec (90^\circ - a) = \sqrt{1 + \tan^2 (90^\circ - a)}$$

hence

$$\operatorname{cosec} a = \sqrt{1 + \cot^2 a}$$

or generally

$$\operatorname{cosec} = \sqrt{1 + \cot^2}$$

Deduction verified by the equation [i]

Because

$$\sec = \frac{1}{\cos}, \text{ by (20)}$$

we have

$$\sec (90^\circ - a) = \frac{1}{\cos (90^\circ - a)}$$

hence

$$\operatorname{cosec} a = \frac{1}{\sin a}$$

or generally,

$$\operatorname{cosec} = \frac{1}{\sin}$$

Deduction verified by the proportion [e]

Because

$$\sec = \frac{\tan}{\sin} \text{ by (21)}$$

we have

$$\sec (90^\circ - a) = \frac{\tan (90^\circ - a)}{\sin (90^\circ - a)}$$

hence

$$\operatorname{cosec} a = \frac{\cot a}{\cos a}$$

or generally $\operatorname{cosec} = \frac{\cot}{\cos}$ [24]

Deduction verified by the proportion [f]

Formulas for Versed Sine and Covered Sine.

By definition (page 10) we have

$$\begin{aligned} \text{vers} &= AM \\ \text{But} \quad AM &= CA - CM \\ \text{Therefore} \quad \text{vers} &= 1 - \cos \end{aligned} \quad (25)$$

$$\begin{aligned} \text{Because} \quad \text{vers} &= 1 - \cos, \text{ by (25)} \\ \text{we have} \quad \text{vers } (90^\circ - a) &= 1 - \cos (90^\circ - a) \\ \text{hence} \quad \text{coveis} &= 1 - \sin a \\ \text{or generally} \quad \text{coveis} &= 1 - \sin \end{aligned} \quad (26)$$

Besides the above formulae, many others may be obtained by combining them. The most usually employed are the following;
Substituting (19) in (2) gives

$$\sin = \frac{\tan}{\sqrt{1 + \tan^2}} \quad (27)$$

Substituting (18) in (3) gives

$$\sin = \frac{\cos}{\sqrt{\operatorname{cosec}^2 - 1}} \quad (28)$$

Substituting (22) in (5) gives

$$\sin = \frac{1}{\sqrt{1 + \cot^2}} \quad (29)$$

Substituting (22) in (7) gives

$$\cos = \frac{\cot}{\sqrt{1 + \cot^2}} \quad (30)$$

Substituting (14) in (8) gives

$$\cos = \frac{\sin}{\sqrt{\sec^2 - 1}} \quad (31)$$

[24]

Substituting (19) in (10) gives

$$\cos = \frac{1}{\sqrt{1+\tan^2}} \quad (32)$$

Substituting (6) in (11) gives

$$\tan = \frac{\sin}{\sqrt{1-\sin^2}} \quad (33)$$

(25)

Substituting (1) in (11) gives

$$\tan = \frac{\sqrt{1-\cos^2}}{\cos} \quad (34)$$

(26)

Substituting (1) and (6) in (11) gives

$$\tan = \frac{\sqrt{1-\cos^2}}{\sqrt{1-\sin^2}} \quad (35)$$

ained by com-
lowing;

(27)

Substituting (17) in (12) gives

$$\tan = \frac{1}{\cos \times \operatorname{cosec}} \quad (36)$$

(28)

Substituting (1) in (15) gives

$$\cot = \frac{\cos}{\sqrt{1-\cos^2}} \quad (37)$$

(29)

Substituting (6) in (15) gives

$$\cot = \frac{\sqrt{1-\sin^2}}{\sin} \quad (38)$$

(30)

Substituting (6) and (1) in (15) gives

$$\cot = \frac{\sqrt{1-\sin^2}}{\sqrt{1-\cos^2}} \quad (39)$$

(31)

Substituting (13) in (16) gives

$$\cot = \frac{1}{\sin \times \sec} \quad (40)$$

Substituting (6) in (20) gives

$$\sec = \frac{1}{\sqrt{1 - \sin^2}} \quad (41)$$

Substituting (9) in (20) gives

$$\sec = \frac{1}{\sin \times \cot} \quad (42)$$

Substituting (1) in (21) gives

$$\sec = \frac{\tan}{\sqrt{1 - \cos^2}} \quad (43)$$

Substituting (1) in (23) gives

$$\operatorname{cosec} = \frac{1}{\sqrt{1 - \cos^2}} \quad (44)$$

Substituting (4) in (23) gives

$$\operatorname{cosec} = \frac{1}{\tan \times \cos} \quad (45)$$

Substituting (6) in (24) gives

$$\operatorname{cosec} = \frac{\cot}{\sqrt{1 - \sin^2}} \quad (46)$$

By the above deductions we see that Sine and Cosecant are reciprocal to each other, as are, also, Cosine and Secant, and Tangent and Cotangent; so that we may write

$$\sin \times \operatorname{Cosec} = 1$$

$$\cos \times \sec = 1$$

$$\tan \times \cot = 1$$

The above results being of the highest importance in all fundamental investigations, are collected and arranged in the following table, in which a denotes any angle or arc. •

TABLE I.

FUNDAMENTAL FORMULAS.

(41)

$$\sin a = \sqrt{1 - \cos^2 a} \quad \cos a = \sqrt{1 - \sin^2 a}$$

(42)

$$\begin{aligned} &= \frac{\tan a}{\sec a} &= \frac{\cot a}{\operatorname{cosec} a} \\ &= \frac{\cos a}{\cot a} &= \frac{\sin a}{\tan a} \end{aligned}$$

(43)

$$\begin{aligned} &= \cos a \tan a &= \sin a \cot a \\ &= \frac{1}{\operatorname{cosec} a} &= \frac{1}{\sec a} \end{aligned}$$

(44)

$$\begin{aligned} &= \frac{\tan a}{\sqrt{1 + \tan^2 a}} &= \frac{\cot a}{\sqrt{1 + \cot^2 a}} \\ &= \frac{\cos a}{\sqrt{\sec^2 a - 1}} &= \frac{\sin a}{\sqrt{\sec^2 a - 1}} \end{aligned}$$

(45)

$$\begin{aligned} &= \frac{1}{\sqrt{1 + \cot^2 a}} &= \frac{1}{\sqrt{1 + \tan^2 a}} \end{aligned}$$

$$\begin{aligned} \tan a &= \frac{\sin a}{\cos a} & \cot a &= \frac{\cos a}{\sin a} \\ &= \frac{1}{\cot a} & &= \frac{1}{\tan a} \end{aligned}$$

(46)

$$\begin{aligned} &= \sin a \sec a &= \cos a \operatorname{cosec} a \\ &= \sqrt{\sec^2 a - 1} &= \sqrt{\operatorname{cosec}^2 a - 1} \end{aligned}$$

Cosecant are re-
sult, and Tangent

$$\begin{aligned} &= \frac{\sin a}{\sqrt{1 - \sin^2 a}} &= \frac{\cos a}{\sqrt{1 - \cos^2 a}} \\ &= \frac{\sqrt{1 - \cos^2 a}}{\cos a} &= \frac{\sqrt{1 - \sin^2 a}}{\sin a} \end{aligned}$$

ce in all funda-
n the following

$$\begin{aligned} &= \sqrt{\frac{1 - \cos^2 a}{1 - \sin^2 a}} &= \sqrt{\frac{1 - \sin^2 a}{1 - \cos^2 a}} \\ &= \frac{1}{\cos a \operatorname{cosec} a} &= \frac{1}{\sin a \sec a} \end{aligned}$$

$$\begin{aligned}
 \sec a &= \sqrt{[1 + \tan^2 a]} & \operatorname{cosec} a &= \sqrt{[1 + \cot^2 a]} \\
 &= \frac{1}{\cos a} & &= \frac{1}{\sin a} \\
 &= \frac{\tan a}{\sin a} & &= \frac{\cot a}{\cos a} \\
 &= \frac{1}{\sqrt{[1 - \sin^2 a]}} & &= \frac{1}{\sqrt{[1 - \cos^2 a]}} \\
 &= \frac{1}{\sin a \cot a} & &= \frac{1}{\cos a \tan a} \\
 &= \frac{\tan a}{\sqrt{[1 - \cos^2 a]}} & &= \frac{\cot a}{\sqrt{[1 - \sin^2 a]}}
 \end{aligned}$$

$$\operatorname{Vers} a = 1 - \cos a \quad \operatorname{Covers} a = 1 - \sin a$$

CHAPTER III.

VARIATION OF ANGULAR FUNCTIONS.

By the definition of a *function* given in Chapter I, [page 8], it is obvious that the term function implies variability, or that two or more quantities vary together in accordance with some mathematical law.

Any one of two quantities connected by a functional relation, may be assumed to vary arbitrarily; it is then called the *independent variable*, whilst the other, which varies dependently of the former, is called the *dependent variable*. For the latter, the term *function* is generally reserved.

In angular functions, the independent variable is the angular magnitude; and the trigonometric quantities relative to that angle or to the arc measuring that angle, are the dependent variables.

Hitherto we have established certain relations between angular functions independently of their variation in value. We shall next

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proceed to investigate the principles which rule this variation.

Angular functions vary in two ways: 1st in their *relative* value, or in respect to algebraic signs; 2nd in their *absolute* or *numerical* value:

Algebraic signs of Angular Functions.

The Algebraic Symbols $+$ and $-$ are not only indicative of the operations of addition and subtraction, as defined in the beginning of elementary works on Algebra, but are, also, indicative of the *qualities* of quantities. And, far from confusion arising from this double mode of interpreting algebraic symbols, analysis, on the contrary gains thereby considerably in power.

Let C be a fixed point on a fixed straight line AA' . Then to

$A' \quad C \quad A$

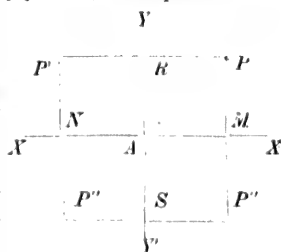
determine the position of any point on that line, we must know the distance of that point from C and, also, *on which side of C the point lies*. These two opposite directions are respectively indicated by the signs $+$ and $-$ so that, if the distances measured from C towards the right hand be denoted by $+$, the distances towards the left hand will be denoted by $-$; thus, if A and A' are both distant 4 from C , the distance of A is denoted by $+4$, and that of A' by -4 .

Again, let XX' and YY' be the two fixed straight lines which meet at right angle at A .

To determine the position of any point P , in the plane containing the two lines, we ought to

know the distance of P from each of these lines, and, also, on which side of each it lies. The two opposite directions respecting each of the lines XX' and YY' , are distinguished by the signs $+$ and $-$, as before; so that, if the distances measured from XX' in the direction *above* be indicated by $+$

those measured from the same straight line in the direction *below*



are indicated by —; and the distances measured from YY' towards the right hand being marked by +, those towards the left hand are designated by —. Thus, if P lies in the first quadrant XAY , both distances PM and PR are positive; for P' lying in the second quadrant XAY' we have $P'N$ positive and $P'R$ negative; for P'' lying in the third $X'AY'$, the both distances $P''N$ and $P''S$ are negative; and for P''' lying in the fourth quadrant XAY'' , we have, $P'''M$ negative, and $P'''S$ positive.

On this convention rests the principle that rules the signs of Trigonometric lines.

All lines perpendicular on the diameter AA' , and lying above are considered as positive, or have the sign +; and those lying below have the sign —.

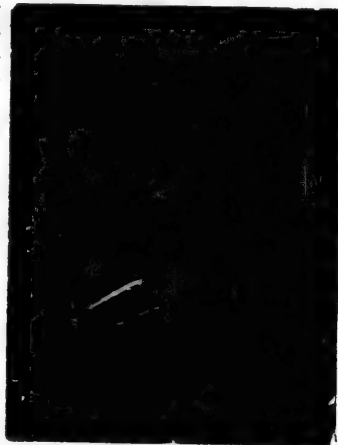
All lines perpendicular on the diameter BB' and extending towards the right, are considered as positive, or have the sign +; and those extending towards the left have the sign —.

Thus it is seen by the inspection of the figure that for an arc AP_1 not exceeding the first quadrant or 90° , both the sine P_1M_1 , and the cosine P_1L_1 , or its equal CM_1 , are positive.

In the second quadrant, the sine P_2M_2 is positive, and the cosine P_2L_2 , or its equal CM_2 , is negative.

In the third quadrant, both the sine P_3M_3 , and the cosine P_3L_3 , or its equal CM_3 , are negative.

In the fourth quadrant, the sine P_4M_4 is negative; and the cosine P_4L_4 , or its



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equal OM , positive, that is, supposing a to denote any number of degrees less than 90 ; we have

$$+\sin(90^\circ - a) \quad (47) \quad +\cos(90^\circ - a) \quad (51)$$

$$+\sin(90^\circ + a) \quad (48) \quad -\cos(90^\circ + a) \quad (52)$$

$$-\sin(180^\circ + a) \quad (49) \quad -\cos(180^\circ + a) \quad (53)$$

$$-\sin(270^\circ + a) \quad (50) \quad +\cos(270^\circ + a) \quad (54)$$

The signs of the sine and cosine being determined, the signs of all the other Trigonometric lines may be at once established by substituting the above values in (11), (15), (20), and (23), Thus:

$$\frac{+\sin(90^\circ - a)}{+\cos(90^\circ - a)} = +\tan(90^\circ - a) \quad (55)$$

$$\frac{+\sin(90^\circ + a)}{-\cos(90^\circ + a)} = -\tan(90^\circ + a) \quad (56)$$

$$\frac{-\sin(180^\circ + a)}{-\cos(180^\circ + a)} = +\tan(180^\circ + a) \quad (57)$$

$$\frac{-\sin(270^\circ + a)}{+\cos(270^\circ + a)} = -\tan(270^\circ + a) \quad (58)$$

$$\frac{+\cos(90^\circ - a)}{+\sin(90^\circ - a)} = +\cot(90^\circ - a) \quad (59)$$

$$\frac{-\cos(90^\circ + a)}{+\sin(90^\circ + a)} = -\cot(90^\circ + a) \quad (60)$$

$$\frac{-\cos(180^\circ + a)}{-\sin(180^\circ + a)} = +\cot(180^\circ + a) \quad (61)$$

$$\frac{+\cos (270^{\circ}+a)}{-\sin (270^{\circ}+a)}=-\cot (270^{\circ}+a) \quad (62)$$

$$\frac{1}{+\cos (90^{\circ}-a)}=+\sec (90^{\circ}-a) \quad (63)$$

$$\frac{1}{-\cos (90^{\circ}+a)}=-\sec (90^{\circ}+a) \quad (64)$$

$$\frac{1}{-\cos (180^{\circ}+a)}=-\sec (180^{\circ}+a) \quad (65)$$

$$\frac{1}{+\cos (270^{\circ}+a)}=+\sec (270^{\circ}+a) \quad (66)$$

$$\frac{1}{+\sin (90^{\circ}-a)}=+\operatorname{cosec} (90^{\circ}-a) \quad (67)$$

$$\frac{1}{+\sin (90^{\circ}+a)}=+\operatorname{cosec} (90^{\circ}+a) \quad (68)$$

$$\frac{1}{-\sin (180^{\circ}+a)}=-\operatorname{cosec} (180^{\circ}+a) \quad (69)$$

$$\frac{1}{-\sin (270^{\circ}+a)}=-\operatorname{cosec} (270^{\circ}+a) \quad (70)$$

Hence it is seen that Tangent and Cotangent have the same sign in each quadrant; and both are positive when Sine and Cosine have alike sign, and negative when they have unlike sign.

Secant has always the same sign as Cosine; and Cosecant has always the same sign as Sine.

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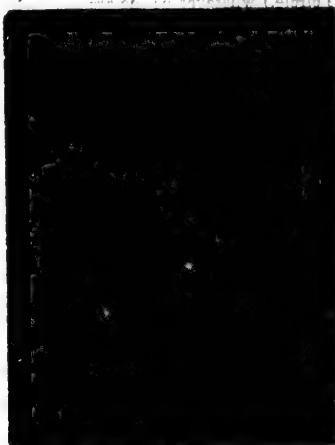
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Versed sine and Covered sine are positive in the four quadrants because, as seen in (45) and (46), they are always equal to the radius, or unit, diminished respectively by Cosine and sine, which never exceed unity.

These analytical results may be verified by the inspection of the figure below, where we have, for the arc AP , not exceeding the first quadrant,



Tangent AE_1 , *positive*, being above AA_1 ,
Cotangent BR_1 , *positive*, being to the right of BB_1 ,

And for the arc AP_2 in the second quadrant:

Tangent AE_2 , *negative*, being below AA_1 ,
Cotangent BR_2 , *negative*, being to the left of BB_1 ,

And for the arc AA_1P_3 in the third quadrant:

Tangent AE_3 , *positive*, being above AA_1 ,
Cotangent BR_3 , *positive*, being to the right BB_1 ,

And for the arc AA_1P_4 in the fourth quadrant:

Tangent AE , *negative*, being below AA_1 ,
Cotangent BR , *negative*, being to the left of BB_1 ,

Where we see a perfect agreement with the results (55), (56) (57) (58), (59), (60), (61), and (62).

Secant and Cosecant are estimated from one and the same point (the centre of the circle,) whatever be the arc they are relating to; therefore, each of them has not its opposite directions referred to the diameters AA_1 and BB_1 , but to the extremity of the arc, so that they are considered positive when estimated towards the end of the arc, and negative when estimated in an opposite direction. Thus we have for the arc AP , in the first quadrant:

Secant CE , *positive*;

Cosecant CR , *positive*;

And for the arc AP , in the second quadrant:

Secant CE , *negative*;

Cosecant CR , *positive*;

And for the arc AP , in the third quadrant:

Secant CE , *positive*;

Cosecant CR , *negative*;

And for the arc AP , in the fourth quadrant:

Secant CE , *negative*;

Cosecant CR , *negative*;

which all agree with (63), (64), (65), (66), (67), (68), (69), and (70).

Versed Sine and Covered Sine being always estimated in the same direction, the first from A towards A_1 , and the second from B towards B_1 , must not change in signs, and therefore remain positive in the four quadrants.

For the sake of reference, the relative variations of Trigonometric lines, as above established, are exhibited in the following table.

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TABLE II.

RELATIVE VARIATIONS OF ANGULAR FUNCTIONS,

	1. Quad.	2. Quad.	3. Quad.	4. Quad.
Sin	+	+	—	—
Cos	+	—	—	+
Tan	+	—	+	—
Cot	+	—	+	—
Sec	+	—	—	+
Cosec	+	+	—	—
Vers	+	+	+	+
Covers	+	+	+	+

, (69), and (70).

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following table.

Numerical Variations of Angular Functions.

The variable angle, or the angle of different magnitudes, which angular functions are referred to, may be conceived as generated by the revolution of a movable radius CP round C , from the fixed radius AC , in a direction to the left. It is manifest that as the angle decreases, the sine, also, decreases, because P_1M_1 is less than P_2M_2 , ...

When the revolving radius coincides with CA , or the angle becomes 0 , $PM = 0$. Therefore we have

$$\sin 0^\circ = 0 \quad (71)$$

When the revolving radius coincides with CB , or the angle becomes ACB , a right angle, PM also coincides with CB , and is equal to the radius = 1. Therefore, we have

$$\sin 90^\circ = 1 \quad (72)$$

Again, when the revolving radius proceeds in the second quadrant, the sine diminishes as the angle increases, because P_1M_1 is greater than P_2M_2 , ...

When the revolving radius coincides with CA , or the angle is equal to two right angles, $PM = 0$. Therefore we have

$$\sin 180^\circ = 0 \quad (73)$$

Reasoning in the same manner for the third and fourth quadrants, we shall find

$$\sin 270^\circ = -1 \quad (74)$$

$$\sin 360^\circ = \sin 0^\circ = 0 \quad (75)$$

If now we observe the variations of cosine, we see evidently that as the angle increases, the cosine decreases in the first quadrant, increases in the second, decreases in the third, and increases in the fourth, and we have

$$\cos 0^\circ = CA = 1 \quad (76)$$

$$\cos 90^\circ = 0 \quad (77)$$

$$\cos 180^\circ = -1 \quad (78)$$

$$\cos 270^\circ = 0 \quad (79)$$

$$\cos 360^\circ = \cos 0 = 1 \quad (80)$$

The negative sign is employed for $\sin 270^\circ$ and $\cos 180^\circ$, be-

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cause CB , is below AA , and CA , is to the left of BB .

Hence it is seen, first, that sine

increases with $+$ from 0 to 1, in the 1st quadrant;

decreases with $+$ from 1 to 0 in the 2nd. "

increases with $-$ from 0 to 1, in the 3rd. "

decreases with $-$ from 1 to 0, in the 4th. "

And that conversely Cosine

decreases with $+$ from 1 to 0, in the 1st quadrant;

increases with $-$ from 0 to 1, in the 2nd. "

decreases with $-$ from 1 to 0, in the 3rd. "

increases with $+$ from 0 to 1, in the 4th. "

It is seen, secondly, that the limits of the variations being the same, 0 and 1, for both Sine and Cosine in each of the four quadrants, it follows that the values of Sine and Cosine of all arcs extending to any quadrant are included between 0 and 1, therefore, the numerical value of Sine and Cosine of any angle greater than 90° , is reducible to either Sine or Cosine of any angle not exceeding 90° .

This most important conclusion is illustrated in the following analysis:

Because Sine increases from 0 to 1 between 0° and 90° , in the same proportion as Cosine decreases from 1 to 0 in the same quadrant, it follows that:

$$\sin 0^\circ = \cos 90^\circ$$

$$\sin 1^\circ = \cos 89^\circ$$

$$\sin 2^\circ = \cos 88^\circ \dots\dots$$

$$\text{and generally, } \sin a = \cos (90^\circ - a) \quad (81)$$

For the same reason, we have

$$\cos a = \sin (90^\circ - a) \quad (82)$$

that is; *the sine of an angle is equal to the cosine of its complement; and the cosine of an angle, is equal to the sine of its complement.*

Because both Sine and Cosine decrease in the same proportion from 1 to 0, the former between 90° and 180° , the latter between 0° and 90° , it follows that

$$\sin (90^\circ) = \cos 0^\circ$$

$$\begin{aligned}\sin (90^{\circ}+1) &= \cos 1^{\circ} \\ \sin (90^{\circ}+2) &= \cos 2^{\circ} \dots\dots\dots \\ \text{and generally, } \sin (90^{\circ}+a) &= \cos a\end{aligned}\tag{83}$$

In the same way we obtain

$$\cos (90^{\circ}+a) = -\sin a\tag{84}$$

Because $\cos a = \sin (90^{\circ}+a)$, by (83)

and $\cos a = \sin (90^{\circ}-a)$, by (82)

then $\sin (90^{\circ}+a) = \sin (90^{\circ}-a)$

but $\sin (90^{\circ}+a) = \sin [180^{\circ}-(90^{\circ}-a)]$, which is evident,

therefore, $\sin [180^{\circ}-(90^{\circ}-a)] = \sin (90^{\circ}-a)$

Substituting in these two supplementary angles a for $(90^{\circ}-a)$ the supplemental relation remain the same, and we have

$$\sin (180^{\circ}-a) = \sin a\tag{85}$$

that is, *the sine of an angle is equal to the sine of its supplement.*

Because $\sin a = \cos (90^{\circ}+a)$, by (84)

and $\sin a = \cos (90^{\circ}-a)$, by (81)

we have $-\cos (90^{\circ}+a) = \cos (90^{\circ}-a)$

but $-\cos (90^{\circ}+a) = -\cos [180^{\circ}-(90^{\circ}-a)]$

Therefore $-\cos [180^{\circ}-(90^{\circ}-a)] = \cos (90^{\circ}-a)$

Hence $-\cos (180^{\circ}-a) = \cos a$

Or $\cos (180^{\circ}-a) = -\cos a\tag{86}$

That is, the cosine of an angle and the cosine of its supplement are numerically equal, but have unlike sign.

Since sine increases between 180° and 270° in the same proportion as between 0° and 90° , it follows that

$$-\sin 180^{\circ} = \sin 0^{\circ}$$

$$-\sin 181^{\circ} = \sin 1^{\circ}$$

$$-\sin 182^{\circ} = \sin 2^{\circ} \dots\dots\dots$$

and generally, $-\sin (180^{\circ}+a) = \sin a$

or $\sin (180^{\circ}+a) = -\sin a\tag{87}$

And for the same reason, we have

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$$\begin{aligned} - \cos(180^\circ + a) &= \cos a \\ \text{or} \quad \cos(180^\circ + a) &= -\cos a \end{aligned} \quad (88)$$

(83) Because $\sin a = -\sin(180^\circ + a)$, by (87)
and $\sin a = \sin(180^\circ - a)$, by (85)

(84) we have $\sin(180^\circ - a) = -\sin(180^\circ + a)$

Hence $\sin(180^\circ + 90^\circ - a) = -\sin(180^\circ - 90^\circ + a)^*$

or $\sin(270^\circ - a) = -\sin(90^\circ + a)$

but $\sin(90^\circ + a) = \cos a$, by (83)

Therefore $\sin(270^\circ - a) = -\cos a \quad (89)$

* This inference rests on the principle that when the value of one quadrant is added or subtracted from two angles, the relation of equality which was found existing between the numerical values of some functions of the primitive angles subsists between the remainders. For assume the arcs $AP_1, BP_1, BP_1, AP_1, AP_1, BP_1, BP_1, AP_1$ to be all equal to one another. Then it is easy to prove geometrically, by the equality of triangles, that

$$\begin{aligned} P_1M_1 &= P_1M_4 = P_1M_3 = P_1M_2 \\ P_1M_2 &= P_1M_3 = P_1M_4 = P_1M_1 \end{aligned}$$

and But P_1M_1 is the sine of the arc AA_1P_1 and P_1M_2 is the sine of the arc AP_1 therefore we have $\sin AA_1P_1 = \sin AP_1$

Subtracting from AA_1P_1 the quadrant P_1P_1 leaves the arc AP_1 whose sine is P_1M_1

Subtracting from AP_1 the quadrant P_1P_1 leaves the arc AP_1 whose sine is P_1M_1 .

And since $P_1M_1 = P_1M_2$, we have

$$\sin AP_1 = \sin AP_1$$

that is, if $\sin AA_1P_1 = \sin AP_1$

we have $\sin(AA_1P_1 - 90^\circ) = \sin(AP_1 - 90^\circ)$

This holds true, if, instead of diminishing AA_1P_1 by the quadrant P_1P_1 , we increase it by the quadrant P_1P_1 ; for $P_1M_1 = P_1M_4 = P_1M_1$

Again since $P_1M_1 = P_1M_3$, we have

$$\sin AA_1P_1 = \sin AA_1P_1$$

From AA_1P_1 subtract the quadrant P_1P_1 and from AA_1P_1 subtract the quadrant P_1P_1 , there remains AA_1P_1 whose sine is P_1M_2 ; and AP_1 whose sine is P_1M_4 . But

$$P_1M_2 = P_1M_4 \text{ hence } \dots$$

The same holds true, if, instead of diminishing AA_1P_1 by the quadrant P_1P_1 , we increase it by P_1P_1 for $P_1M_1 = P_1M_4 = P_1M_4$, and the principle is general for all cases. The same result is obtained by adding or subtracting two quadrants, or any other multiple of full quadrants, as is evident.

It must be borne in mind that the equality relates only to the numerical value, and not to the algebraic signs, which change according to the law that rules the relative variations, as before established.



By following a similar process we obtain

$$-\cos (270^\circ - a) = \sin a$$

$$\text{or} \quad \cos (270^\circ - a) = -\sin a \quad (90)$$

Since both Sine and Cosine decrease from 1 to 0, in the same proportion, the former between 270° and 360° , and the latter between 0° and 90° , it follows, observing the rule for the Signs, that

$$-\sin 270^\circ = \cos 0^\circ$$

$$-\sin 271^\circ = \cos 1^\circ$$

$$-\sin 272^\circ = \cos 2^\circ \dots \dots$$

$$\text{and generally } -\sin (270^\circ + a) = \cos a$$

$$\sin (270^\circ + a) = -\cos a \quad (91)$$

And for the same reason we have

$$\cos (270^\circ + a) = \sin a \quad (89)$$

$$\text{Because} \quad \cos a = -\sin (270^\circ + a), \text{ by (91)}$$

$$\text{and} \quad \cos a = -\sin (270^\circ - a), \text{ by (89)}$$

$$\text{we have} \quad -\sin (270^\circ - a) = -\sin (270^\circ + a)$$

$$\text{hence} \quad -\sin (360^\circ - a) = \sin (180^\circ + a), \text{ see page 33}$$

$$\text{but} \quad -\sin (180^\circ + a) = \sin a, \text{ by (87)}$$

$$\text{therefore} \quad -\sin (360^\circ - a) = \sin a$$

$$\text{or} \quad \sin (360^\circ - a) = -\sin a \quad (93)$$

By following a similar process, we obtain, observing the convention for the signs,

$$\cos (360^\circ - a) = \cos a \quad (94)$$

Since the movable radius, after running through 360° , recommences the same revolution as before, we have,

$$\sin (360^\circ + a) = \sin a \quad (95)$$

$$\cos (360^\circ + a) = \cos a \quad (96)$$

that is the Sine and Cosine of an angle which exceeds 360° are the same as those of the excess above 360° .

The sines being equal, so will be the cosines, since

$$\cos = \sqrt{1 - \sin^2}$$

The sines and cosines being equal, so will be all the trigonometric lines, since they are expressed in terms of sine and cosine.

It is evident that what is said here of the arcs is applicable to the angle which the arcs measure.

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The important deductions which precede may be verified geometrically, for the satisfaction of the student, by the annexed figure

where the arc AP_1 is made equal to a ; and the arcs $BP_1, BP_2, A_1P_1, A_1P_2, B_1P_1, B_1P_2, AP_3$ are all supposed equal to $P = a$

Therefore, by construction, the triangles $ACP_1, BCP_1, BCP_2, A_1CP_1, A_1CP_2, B_1CP_1, B_1CP_2, ACP_3$ are all equal. Hence, observing the convention for the algebraic signs, we have

$$P_1M_2 = CM_1$$

or $\sin AP_2 = \cos AP_1$

$$\text{but } AP_2 = AB - BP_2 = 90^\circ - a$$

$$\text{therefore } \sin 90^\circ - a = \cos a \quad (82)$$

Also P_2D_1 , or its equal $CM_2 = P_1M_1$

$$\text{or } \cos AP_2 = \sin AP_1$$

$$\text{but } AP_2 = AB - BP_2 = 90^\circ - a$$

$$\text{therefore } \cos (90^\circ - a) = \sin a \quad (81)$$

$$\text{Also } P_1M_3 = CM_1$$

$$\text{or } \sin AP_3 = \cos AP_1$$

$$\text{but } AP_3 = AB + BP_3 = 90^\circ + a$$

$$\text{therefore } \sin (90^\circ + a) = \cos a \quad (83)$$

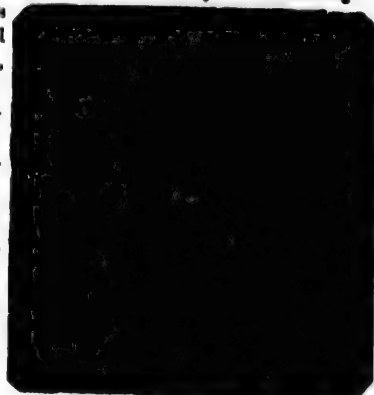
Also $-P_3D_1$ or its equal $CM_3 = P_1M_1$

$$\text{or } -\cos AP_3 = \sin AP_1$$

$$\text{therefore } -\cos (90^\circ + a) = \sin a$$

$$\text{or } \cos (90^\circ + a) = -\sin a \quad (84)$$

$$\text{Also } P_4M_4 = P_1M_1$$



or $\sin AP_4 = \sin AP_1$
 but $AP_4 = ABA_1 - A_1P_4 = 180^\circ - a$
 therefore $\sin (180^\circ - a) = \sin a$ (85)

Also $CM_4 = CM_1$
 or $-\cos AP_4 = \cos AP_1$
 therefore $-\cos (180^\circ - a) = \cos a$
 or $\cos (180^\circ - a) = -\cos a$ (86)

Also $P_4M_4 = P_1M_1$
 or $-\sin AP_4 = \sin AP_1$
 but $AP_4 = ABA_1 + A_1P_4 = 180^\circ + a$
 therefore $-\sin (180^\circ + a) = \sin a$
 or $\sin (180^\circ + a) = -\sin a$ (87)

Also $CM_4 = CM_1$
 that is $-\cos (180^\circ + a) = \cos a$
 or $\cos (180^\circ + a) = -\cos a$ (88)

Also $P_4M_4 = CM_1$
 or $-\sin AP_4 = \cos AP_1$
 but $\sin AP_4 = AA_1B_1 - B_1P_4 = 270^\circ - a$
 therefore $-\sin (270^\circ - a) = \cos a$
 or $\sin (270^\circ - a) = -\cos a$ (89)

Also P_4D_4 , or its equal $CM_4 = P_1M_1$
 or $-\cos AP_4 = \sin AP_1$
 that is $-\cos (270^\circ - a) = \sin a$
 or $\cos (270^\circ - a) = -\sin a$ (90)

Also $P_4M_4 = CM_1$
 or $-\sin AP_4 = \cos AP_1$
 but $AP_4 = AA_1B_1 + B_1P_4 = 270^\circ + a$
 therefore $-\sin (270^\circ + a) = \cos a$
 or $\sin (270^\circ + a) = -\cos a$ (91)
 Also P_4D_4 , or its equal $CM_4 = P_1M_1$
 that is $\cos (270^\circ + a) = \sin a$

Also
 or
 but
 therefore

Also
 therefore

Substit
 (20) (23)

Also $P_3 M_1 = P_1 M_1$
 or $\sin AA_1 P_3 = \sin AP_1$
 but $AA_1 P_3 = AA_1 B A_1 - AP_3 = 360^\circ - a$
 therefore $\sin (360^\circ - a) = \sin a$ (94)

Also CM_1 is the cosine of both $AA_1 P_3$ and AP_1 ;
 therefore $\cos (360^\circ - a) = \cos a$ (94)

Substituting the above values of Sine and Cosine in (11) (15) (20) (23) gives

$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0 \quad (97)$$

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \infty \quad (98)$$

$$\tan 180^\circ = \frac{\sin 180^\circ}{\cos 180^\circ} = \frac{0}{-1} = 0 \quad (99)$$

$$\tan 270^\circ = \frac{\sin 270^\circ}{\cos 270^\circ} = \frac{-1}{0} = \infty \quad (100)$$

$$\tan 360^\circ = \frac{\sin 360^\circ}{\cos 360^\circ} = \frac{0}{1} = 0 \quad (101)$$

$$\cot 0^\circ = \frac{\cos 0^\circ}{\sin 0^\circ} = \frac{1}{0} = \infty \quad (102)$$

$$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0 \quad (103)$$

$$\cot 180^\circ = \frac{\cos 180^\circ}{\sin 180^\circ} = \frac{-1}{0} = \infty \quad (104)$$

$$\cot 270^\circ = \frac{\cos 270^\circ}{\sin 270^\circ} = \frac{0}{-1} = 0 \quad (105)$$

$$\cot 360^\circ = \frac{\cos 360^\circ}{\sin 360^\circ} = \frac{1}{0} = \infty \quad (106)$$

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1 \quad (107)$$

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} = \infty \quad (108)$$

$$\sec 180^\circ = \frac{1}{\cos 180^\circ} = \frac{1}{-1} = -1 \quad (109)$$

$$\sec 270^\circ = \frac{1}{\cos 270^\circ} = \frac{1}{0} = \infty \quad (110)$$

$$\sec 360^\circ = \frac{1}{\cos 360^\circ} = \frac{1}{1} = 1 \quad (111)$$

$$\operatorname{cosec} 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty \quad (112)$$

$$\operatorname{cosec} 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = 1 \quad (113)$$

$$\operatorname{cosec} 180^\circ = \frac{1}{\sin 180^\circ} = \frac{1}{0} = -\infty \quad (114)$$

$$\operatorname{cosec} 270^\circ = \frac{1}{\sin 270^\circ} = \frac{1}{-1} = -1 \quad (115)$$

$$\operatorname{cosec} 360^\circ = \frac{1}{\sin 360^\circ} = \frac{1}{0} = \infty \quad (116)$$

$$\tan (90^\circ - a) = \frac{\sin (90^\circ - a)}{\cos (90^\circ - a)} = \frac{\cos a}{\sin a} = \cot a \quad (117)$$

$$\tan (90^\circ + a) = \frac{\sin (90^\circ + a)}{\cos (90^\circ + a)} = \frac{\cos a}{-\sin a} = -\cot a \quad (118)$$

$$(107) \quad \tan(180^\circ - a) = \frac{\sin(180^\circ - a)}{\cos(180^\circ - a)} = \frac{\sin a}{-\cos a} = -\tan a \quad (119)$$

$$(108) \quad \tan(180^\circ + a) = \frac{\sin(180^\circ + a)}{\cos(180^\circ + a)} = \frac{-\sin a}{-\cos a} = \tan a \quad (120)$$

$$(109) \quad \tan(270^\circ - a) = \frac{\sin(270^\circ - a)}{\cos(270^\circ - a)} = \frac{-\cos a}{-\sin a} = \cot a \quad (121)$$

$$(110) \quad \tan(270^\circ + a) = \frac{\sin(270^\circ + a)}{\cos(270^\circ + a)} = \frac{-\cos a}{\sin a} = -\cot a \quad (122)$$

$$(111) \quad \tan(360^\circ - a) = \frac{\sin(360^\circ - a)}{\cos(360^\circ - a)} = \frac{-\sin a}{\cos a} = -\tan a \quad (123)$$

$$(112) \quad \tan(360^\circ + a) = \frac{\sin(360^\circ + a)}{\cos(360^\circ + a)} = \frac{\sin a}{\cos a} = \tan a \quad (124)$$

$$(113) \quad \cot(90^\circ - a) = \frac{\cos(90^\circ - a)}{\sin(90^\circ - a)} = \frac{\sin a}{\cos a} = \tan a \quad (125)$$

$$(114) \quad \cot(90^\circ + a) = \frac{\cos(90^\circ + a)}{\sin(90^\circ + a)} = \frac{-\sin a}{\cos a} = -\tan a \quad (126)$$

$$(115) \quad \cot(180^\circ - a) = \frac{\cos(180^\circ - a)}{\sin(180^\circ - a)} = \frac{-\cos a}{\sin a} = -\cot a \quad (127)$$

$$(116) \quad \cot(180^\circ + a) = \frac{\cos(180^\circ + a)}{\sin(180^\circ + a)} = \frac{-\cos a}{-\sin a} = \cot a \quad (128)$$

$$\cot a \quad (117) \quad \cot(270^\circ - a) = \frac{\cos(270^\circ - a)}{\sin(270^\circ - a)} = \frac{-\sin a}{-\cos a} = \tan a \quad (129)$$

$$= -\cot a \quad (118) \quad \cot(270^\circ + a) = \frac{\cos(270^\circ + a)}{\sin(270^\circ + a)} = \frac{\sin a}{-\cos a} = -\tan a \quad (130)$$

$$\cot (360^\circ - a) = \frac{\cos (360^\circ - a)}{\sin (360^\circ - a)} = \frac{\cos a}{-\sin a} = -\cot a \quad (131)$$

$$\cot (360^\circ + a) = \frac{\cos (360^\circ + a)}{\sin (360^\circ + a)} = \frac{\cos a}{\sin a} = \cot a \quad (132)$$

$$\sec (90^\circ - a) = \frac{1}{\cos (90^\circ - a)} = \frac{1}{\sin a} = \operatorname{cosec} a \quad (133)$$

$$\sec (90^\circ + a) = \frac{1}{\cos (90^\circ + a)} = \frac{1}{-\sin a} = -\operatorname{cosec} a \quad (134)$$

$$\sec (180^\circ - a) = \frac{1}{\cos (180^\circ - a)} = \frac{1}{-\cos a} = -\sec a \quad (135)$$

$$\sec (180^\circ + a) = \frac{1}{\cos (180^\circ + a)} = \frac{1}{-\cos a} = -\sec a \quad (136)$$

$$\sec (270^\circ - a) = \frac{1}{\cos (270^\circ - a)} = \frac{1}{-\sin a} = -\operatorname{cosec} a \quad (137)$$

$$\sec (270^\circ + a) = \frac{1}{\cos (270^\circ + a)} = \frac{1}{\sin a} = \operatorname{cosec} a \quad (138)$$

$$\sec (360^\circ - a) = \frac{1}{\cos (360^\circ - a)} = \frac{1}{\cos a} = \sec a \quad (139)$$

$$\sec (360^\circ + a) = \frac{1}{\cos (360^\circ + a)} = \frac{1}{\cos a} = \sec a \quad (140)$$

$$\operatorname{cosec} (90^\circ - a) = \frac{1}{\sin (90^\circ - a)} = \frac{1}{\cos a} = \sec a \quad (141)$$

$$\operatorname{cosec} (90^\circ + a) = \frac{1}{\sin (90^\circ + a)} = \frac{1}{\cos a} = \sec a \quad (142)$$

$\operatorname{cosec} (180^\circ - a)$

$\operatorname{cosec} (180^\circ + a)$

$\operatorname{cosec} (270^\circ - a)$

$\operatorname{cosec} (270^\circ + a)$

$\operatorname{cosec} (360^\circ - a)$

$\operatorname{cosec} (360^\circ + a)$

In certain cases, the angle to be considered is not a fixed point, but the angle is positive, or negative, or we suppose a revolution in the opposite direction to be considered.

It is manifest that the corresponding angles have corresponding values.

$$\operatorname{cosec}(180^\circ - a) = \frac{1}{\sin(180^\circ - a)} = \frac{1}{\sin a} = \operatorname{cosec} a \quad (143)$$

$$\operatorname{cosec}(180^\circ + a) = \frac{1}{\sin(180^\circ + a)} = \frac{1}{-\sin a} = -\operatorname{cosec} a \quad (144)$$

$$\operatorname{cosec}(270^\circ - a) = \frac{1}{\sin(270^\circ - a)} = \frac{1}{-\cos a} = -\sec a \quad (145)$$

$$\operatorname{cosec}(270^\circ + a) = \frac{1}{\sin(270^\circ + a)} = \frac{1}{-\cos a} = -\sec a \quad (146)$$

$$\operatorname{cosec}(360^\circ - a) = \frac{1}{\sin(360^\circ - a)} = \frac{1}{-\sin a} = -\operatorname{cosec} a \quad (147)$$

$$\operatorname{cosec}(360^\circ + a) = \frac{1}{\sin(360^\circ + a)} = \frac{1}{\sin a} = \operatorname{cosec} a \quad (148)$$

FUNCTIONS OF NEGATIVE ANGLES.

In certain investigations it is convenient to give different signs to the angles themselves. We have hitherto supposed the variable angle to be generated by the revolution of a straight line round a fixed point, in a direction from right to left; and the angles thus formed are considered as positive, or have the sign +, as ACP . If now we suppose an angle to be generated by the revolution of the movable line from CA in the opposite direction, the angle thus formed is considered as negative, or has the sign —, as ACP' .

It is manifest that the functions of an angle considered as negative have the same absolute magnitude as they would have for the corresponding positive angle, or the angle formed by revolving



the movable line in the positive direction until it reaches the position which it has in the figure. There may be found a difference only in the algebraic signs. Thus, suppose the negative angle ACP' to be denoted by a' , and the corresponding positive angle by a ; we have evidently, the angle being in the fourth quadrant, the sine negative and the cosine positive; that is

$$\sin (-a') = -\sin a \quad (149)$$

$$\cos (-a') = \cos a \quad (150)$$

whence by (11) and (15)

$$\tan (-a') = \frac{\sin (-a')}{\cos (-a')} = \frac{-\sin a}{\cos a} = -\tan a \quad (151)$$

$$\cot (-a') = \frac{\cos (-a')}{\sin (-a')} = \frac{\cos a}{-\sin a} = -\cot a \quad (152)$$

$$\sec (-a') = \frac{1}{\cos (-a')} = \frac{1}{\cos a} = \sec a \quad (153)$$

$$\operatorname{cosec} (-a') = \frac{1}{\sin (-a')} = \frac{1}{-\sin a} = -\operatorname{cosec} a \quad (154)$$

Hence, the sine, tangent, cotangent, and cosecant, of a negative angle, have the sign opposite to that of the sine, tangent, cotangent, and cosecant, of the corresponding positive angle: and the cosine and secant have the same sign in both the negative and the corresponding positive.

For the sake of reference, we collect the preceding results in the following table.

2. — Re
Functions

sin (90
sin (90
sin (180
sin (180
sin (270
sin (270
sin (360
sin (360

tan (90
tan (90

TABLE III.

NUMERICAL VARIATIONS OF ANGULAR FUNCTIONS.

1.— *Value of Functions of Quadrantal angles.*

$\sin 0^\circ = 0$	$\cos 0^\circ = 1$
$\sin 90^\circ = 1$	$\cos 90^\circ = 0$
$\sin 180^\circ = 0$	$\cos 180^\circ = -1$
$\sin 270^\circ = -1$	$\cos 270^\circ = 0$
$\sin 360^\circ = 0$	$\cos 360^\circ = 1$
$\tan 0^\circ = 0$	$\cot 0^\circ = \infty$
$\tan 90^\circ = \infty$	$\cot 90^\circ = 0$
$\tan 180^\circ = 0$	$\cot 180^\circ = -\infty$
$\tan 270^\circ = -\infty$	$\cot 270^\circ = 0$
$\tan 360^\circ = 0$	$\cot 360^\circ = \infty$
$\sec 0^\circ = 1$	$\operatorname{cosec} 0^\circ = \infty$
$\sec 90^\circ = \infty$	$\operatorname{cosec} 90^\circ = 1$
$\sec 180^\circ = -1$	$\operatorname{cosec} 180^\circ = \infty$
$\sec 270^\circ = \infty$	$\operatorname{cosec} 270^\circ = -1$
$\sec 360^\circ = 1$	$\operatorname{cosec} 360^\circ = \infty$

2.— *Reduction of the Functions of all Angles to corresponding Functions of an Angle less than 90° .*

$\sin (90^\circ - a) = \cos a$	$\cos (90^\circ - a) = \sin a$
$\sin (90^\circ + a) = \cos a$	$\cos (90^\circ + a) = -\sin a$
$\sin (180^\circ - a) = \sin a$	$\cos (180^\circ - a) = -\cos a$
$\sin (180^\circ + a) = -\sin a$	$\cos (180^\circ + a) = -\cos a$
$\sin (270^\circ - a) = -\cos a$	$\cos (270^\circ - a) = -\sin a$
$\sin (270^\circ + a) = \cos a$	$\cos (270^\circ + a) = \sin a$
$\sin (360^\circ - a) = -\sin a$	$\cos (360^\circ - a) = \cos a$
$\sin (360^\circ + a) = \sin a$	$\cos (360^\circ + a) = \cos a$
$\tan (90^\circ - a) = \cot a$	$\cot (90^\circ - a) = \tan a$
$\tan (90^\circ + a) = -\cot a$	$\cot (90^\circ + a) = -\tan a$

$\tan (180^\circ - a) = -\tan a$	$\cot (180^\circ - a) = -\cot a$
$\tan (180^\circ + a) = \tan a$	$\cot (180^\circ + a) = \cot a$
$\tan (270^\circ - a) = \cot a$	$\cot (270^\circ - a) = \tan a$
$\tan (270^\circ + a) = -\cot a$	$\cot (270^\circ + a) = -\tan a$
$\tan (360^\circ - a) = -\tan a$	$\cot (360^\circ - a) = -\cot a$
$\tan (360^\circ + a) = \tan a$	$\cot (360^\circ + a) = \cot a$
$\sec (90^\circ - a) = \operatorname{cosec} a$	$\operatorname{cosec} (90^\circ - a) = \sec a$
$\sec (90^\circ + a) = -\operatorname{cosec} a$	$\operatorname{cosec} (90^\circ + a) = -\sec a$
$\sec (180^\circ - a) = -\sec a$	$\operatorname{cosec} (180^\circ - a) = \operatorname{cosec} a$
$\sec (180^\circ + a) = \sec a$	$\operatorname{cosec} (180^\circ + a) = -\operatorname{cosec} a$
$\sec (270^\circ - a) = -\operatorname{cosec} a$	$\operatorname{cosec} (270^\circ - a) = \sec a$
$\sec (270^\circ + a) = \operatorname{cosec} a$	$\operatorname{cosec} (270^\circ + a) = -\sec a$
$\sec (360^\circ - a) = \sec a$	$\operatorname{cosec} (360^\circ - a) = -\operatorname{cosec} a$
$\sec (360^\circ + a) = \sec a$	$\operatorname{cosec} (360^\circ + a) = \operatorname{cosec} a$

Thus we see that Sine and Cosine may have any value between -1 and $+1$; Tangent and Cotangent may have any value between 0 and ∞ ; Secant and Cosecant may have any value between 1 and ∞ .

EXERCISES.

1. To determine the value of the angular functions of 45° .

By the equation [a], page 13, we have

$$\sin^2 + \cos^2 = 1$$

but

$$\sin 45^\circ = \cos 45^\circ, \text{ by (82)}$$

therefore

$$\sin^2 45^\circ = \cos^2 45^\circ = \frac{1}{2}$$

hence

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = 0.7071068 \dots$$

and

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = 0.7071068 \dots$$

whence

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$

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$$\cos 30^\circ = \sqrt{\frac{3}{2}}$$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ}$$

$$\cot 30^\circ = \frac{\cos 30^\circ}{\sin 30^\circ}$$

$$\cot 45^\circ = \frac{\cos 45^\circ}{\sin 45^\circ} = 1$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2} = 1.4142136 \dots$$

$$\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2} = 1.4142136 \dots$$

$$\operatorname{vers} 45^\circ = 1 - \cos 45^\circ = 1 - \frac{1}{\sqrt{2}} = 0.2928932 \dots$$

$$\operatorname{covers} 45^\circ = 1 - \sin 45^\circ = 1 - \frac{1}{\sqrt{2}} = 0.2928932 \dots$$

2. To determine the value of Angular Functions of 30° .

Since the sine of an arc is perpendicular to the diameter passing through one extremity of the arc, it follows evidently that the sine of an arc is always equal to half the chord of double the arc, as seen in the figure, page 35, where $P_1 M_1$, the sine of the arc AP_1 , is the half of $P_1 P_2$, the chord of the arc $P_1 A P_2$, double of the arc AP_1 . But it is proved in geometry that the chord of one sixth of the circumference of a circle is equal to the radius of that circle. Therefore, the radius being assumed as unity, we have

$$\sin 30^\circ = \frac{1}{2}, \text{ whence}$$

$$\cos 30^\circ = \sqrt{[1 - \sin^2 30^\circ]} = \sqrt{[1 - \frac{1}{4}]} = \sqrt{\frac{3}{4}} = 0.8660254 \dots$$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = 0.5773502 \dots$$

$$\cot 30^\circ = \frac{\cos 30^\circ}{\sin 30^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = 1.7320508 \dots$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}} = 1.154701\dots$$

$$\operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

$$\operatorname{vers} 30^\circ = 1 - \cos 30^\circ = 1 - \frac{\sqrt{3}}{2} = 0.1339746$$

$$\operatorname{covers} 30^\circ = 1 - \sin 30^\circ = \frac{1}{2}$$

3. To determine the value of Angular Functions of 60° .

Since 60° is the complement of 30° , we have

$$\sin 60^\circ = \cos 30^\circ = 0.8660254\dots$$

$$\cos 60^\circ = \sin 30^\circ = \frac{1}{2}$$

$$\tan 60^\circ = \cot 30^\circ = 1.7320508\dots$$

$$\cot 60^\circ = \tan 30^\circ = 0.5773502\dots$$

and so on.

4. To determine the value of Angular Functions of 120° .

Since $120^\circ = 90^\circ + 30^\circ$

$$\text{then } \sin 120^\circ = \cos 30^\circ = 0.8660254\dots$$

$$\cos 120^\circ = -\sin 30^\circ = -\frac{1}{2}$$

$$\tan 120^\circ = -\cot 30^\circ = -1.7320508\dots$$

$$\cot 120^\circ = -\tan 30^\circ = -0.5773502$$

and so on.

It is manifest that in the same way, we may obtain, at once, the angular functions of 135° , 150° , 210° , 225° , 240° , etc., which are respectively equal to $90^\circ + 45^\circ$, $90^\circ + 60^\circ$, $180^\circ + 30^\circ$, $180^\circ + 45^\circ$, $180^\circ + 60^\circ$...

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of 60° .

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 $0^\circ + 30^\circ, 180^\circ$

CHAPTER IV.

ANGULAR FUNCTIONS INVOLVING MORE THAN ONE VARIABLE.

Fundamental Proposition and Formula.

The sine of the sum of two angles or arcs, is equal to the sum of the product of the sine of the first into the cosine of the second, and the product of the cosine of the first into the sine of the second.

This proposition is called fundamental on account of its great importance, as being, together with the fundamental formulæ given in Chapter II. and relating to functions of a single angle, the base of the whole trigonometrical science.

This proposition will be proved geometrically; and we will next proceed to an analytical investigation.

Let the arc AP
 $= A$; and the arc
 $PD = B$; then
the arc $AD =$
 $AP + PD = A$
 $+ B$. On CA let
fall the perpen-
diculars $FO, TL,$
 DM . Draw the
radius CP ; also
 DT perpendicu-
lar on CP , and



TS perpendicular on *DM*. Then we have, by definition (page 8):

$$PO = \sin A$$

$$CO = \cos A$$

$$DT = \sin B$$

$$CT = \cos B$$

$$DM = \sin AD = \sin (A + B)$$

By construction it is evident that the triangles *CPO*, *CTL*, and *TDS* are similar, therefore; taking the radius as unity, we have

$$CP : PO :: CT : TL$$

$$\text{or } 1 : \sin A :: \cos B : TL \quad [k]$$

$$\text{Also } CP : CO :: DT : DS$$

$$\text{or } 1 : \cos A :: \sin B : DS \quad [l]$$

$$\text{from } [k] \text{ we obtain } TL = \sin A \cos B$$

$$\text{from } [l] \text{ we obtain } DS = \cos A \sin B$$

$$\text{Adding } TL + DS = \sin A \cos B + \cos A \sin B$$

$$\text{but } TL + DS = MS + DS = DM = \sin (A + B)$$

$$\text{Therefore } \sin (A + B) = \sin A \cos B + \cos A \sin B$$

which is the fundamental formula involving two angles.

Angular Functions of (A + B) and (A - B).

Resuming the last formula:

$$\sin (A + B) = \sin A \cos B + \cos A \sin B \quad (155)$$

$$\sin (A - B) = \sin [180^\circ - (A - B)], \text{ by (85)}$$

$$= \sin [(180^\circ - A) + B], \text{ which is evident.}$$

$$= \sin (180^\circ - A) \cos B + \cos (180^\circ - A) \sin B, \text{ by (155)}$$

$$= \sin A \cos B - \cos A \sin B \quad (156)$$

by substituting for $\sin (180^\circ - A)$ and $\cos (180^\circ - A)$ their values as found in (85) and (86).

$$\cos (A + B) = \sin [90^\circ + (A + B)], \text{ by (83)}$$

$$= \sin [(90^\circ + A) + B], \text{ which is evident.}$$

$$= \sin (90^\circ + A) \cos B + \cos (90^\circ + A) \sin B, \quad (155)$$

$$= \cos A \cos B - \sin A \sin B, \text{ by (83) and (84). (175)}$$

$$\begin{aligned} \cos(A - B) &= -\cos[180^\circ - (A - B)], \text{ by (86)} \\ &= -\cos[(180^\circ - A) + B], \text{ which is evident.} \\ &= -[\cos(180^\circ - A) \cos B - \sin(180^\circ - A) \sin B], \\ &\quad \text{by (157)} \\ &= -\cos(180^\circ - A) \cos B + \sin(180^\circ - A) \sin B \\ &= \cos A \cos B + \sin A \sin B, \text{ by (86) and (85). (158)} \end{aligned}$$

$$\begin{aligned} \tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)}, \text{ by (11)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}, \text{ by (155) and (156)} \end{aligned}$$

dividing numerator and denominator by $\cos A \cos B$:

$$\begin{aligned} &\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B} \div \frac{\cos A \cos B - \sin A \sin A}{\cos A \cos B} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \end{aligned} \quad (159)$$

$$\begin{aligned} \tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)}, \text{ by (11)} \\ &= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B} \end{aligned}$$

Proceeding as before.

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad (160)$$

$$\begin{aligned}\cot (A+B) &= \frac{\cos (A+B)}{\sin (A+B)} \text{ by (15)} \\ &= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B} \\ &\quad \text{dividing num. and den. by } \sin A \sin B, \text{ and reduc-} \\ &\quad \text{ing;} \\ &= \frac{\cot A \cot B - 1}{\cot B + \cot A} \quad (161)\end{aligned}$$

$$\begin{aligned}\cot (A-B) &= \frac{\cos (A-B)}{\sin (A-B)}, \text{ by (15)} \\ &= \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B} \\ &\quad \text{Proceeding as before:} \\ &= \frac{\cot A \cot B + 1}{\cot B - \cot A} \quad (162)\end{aligned}$$

$$\begin{aligned}\sec A+B) &= \frac{1}{\cos A+B)}, \text{ by (20)} \\ &= \frac{1}{\cos A \cos B - \sin A \sin B}\end{aligned}$$

Multiplying num. and den. by $\sec A \sec B$:

$$= \frac{\sec A \sec B}{1 - \tan A \tan B} \quad (163)$$

Multiplying num. and den. of (163) by $\operatorname{cosec} A \operatorname{cosec} B$, and observing that by (10) and (36). $\tan \times \operatorname{cosec} = \sec$:

$$= \frac{\sec A \sec B \operatorname{cosec} A \operatorname{cosec} B}{\operatorname{cosec} A \operatorname{cosec} B - \sec A \sec B} \quad (164)$$

$$\sec (A - B) = \frac{1}{\cos (A - B)}$$

$$= \frac{1}{\cos A \cos B + \sin A \sin B}$$

Proceeding as before:

$$= \frac{\sec A \sec B}{1 + \tan A \tan B} \quad (165)$$

$$= \frac{\sec A \sec B \operatorname{cosec} A \operatorname{cosec} B}{\operatorname{cosec} A \operatorname{cosec} B + \sec A \sec B} \quad (166)$$

$$\operatorname{cosec} (A + B) = \frac{1}{\sin (A + B)}, \text{ by (23)}$$

$$= \frac{1}{\sin A \cos B + \cos A \sin B}$$

Multiplying num. and den. by $\operatorname{cosec} A \sec B$, and reducing:

$$= \frac{\operatorname{cosec} A \sec B}{1 + \cot A \tan B} \quad (167)$$

Multiplying num. and den. of (167) by $\sec A \operatorname{cosec} B$, and observing that, by (5) and (40), $\cot \times \sec = \operatorname{cosec}$:

$$= \frac{\sec A \sec B \operatorname{cosec} A \operatorname{cosec} B}{\sec A \operatorname{cosec} B + \sec B \operatorname{cosec} A} \quad (168)$$

$$\operatorname{cosec} (A - B) = \frac{1}{\sin (A - B)}, \text{ by (23)}$$

$$= \frac{1}{\sin A \cos B - \cos A \sin B}$$

Proceeding as before:

$$= \frac{\operatorname{cosec} A \sec B}{1 - \cot A \tan B} \quad (169)$$

$$= \frac{\sec A \sec B \operatorname{cosec} A \operatorname{cosec} B}{\sec A \operatorname{cosec} B - \sec B \operatorname{cosec} A} \quad (170)$$

Particular Case of 45°

Let $A = 45^\circ$; then we obtain:

$$\sin (45^\circ + B) = \frac{\cos B + \sin B}{\sqrt{2}} \quad (171)$$

$$\cos (45^\circ + B) = \frac{\cos B - \sin B}{\sqrt{2}} \quad (172)$$

$$\tan (45^\circ + B) = \frac{\cos B + \sin B}{\cos B - \sin B}$$

Dividing num. and den. by $\cos B$ or $\sin B$:

$$= \frac{1 + \tan B}{1 - \tan B} \quad (173)$$

$$= \frac{\cot B + 1}{\cot B - 1} \quad (174)$$

$$\cot (45^\circ + B) = \frac{\cos B - \sin B}{\cos B + \sin B}$$

$$= \frac{\cot B - 1}{\cot B + 1} \quad (176)$$

$$\sec (45^\circ + B) = \frac{\sqrt{2}}{\cos B - \sin B} \quad (177)$$

$$\operatorname{cosec} (45^\circ + B) = \frac{\sqrt{2}}{\cos B + \sin B} \quad (178)$$

(169)

Since $(45^\circ + B)$ and $(45^\circ - B)$ are complementary, we have:

(170)

$$\sin (45^\circ - B) = \cos (45^\circ + B) = (172)$$

$$\cos (45^\circ - B) = \sin (45^\circ + B) = (171)$$

$$\tan (45^\circ - B) = \cot (45^\circ + B) = (175) \text{ or } (176)$$

And so on.

Angular Functions of $2A$

(171)

Making $B = A$ in the expressions for $(A + B)$, we obtain:

(172)

$$\sin 2A = \sin (A + A) = \sin A \cos A + \cos A \sin A \quad (179)$$

$$= 2 \sin A \cos A \quad (180)$$

$$= 2 \sin A \sqrt{1 - \sin^2 A} \quad (181)$$

$$= 2 \cos A \sqrt{1 - \cos^2 A} \quad (181)$$

or $\sin B$:

$$\cos 2A = \cos (A + A) \quad (182)$$

$$= \cos^2 A - \sin^2 A \quad (182)$$

$$= 2 \cos^2 A - 1 \quad (183)$$

$$= 1 - 2 \sin^2 A \quad (184)$$

(173)

(174)

$$\text{also} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}, \text{ see equation [a], page 13.}$$

Dividing num. and den. by $\cos^2 A$:

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A} \quad (185)$$

$$\begin{aligned} \tan 2A &= \tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} \\ &= \frac{2 \tan A}{1 - \tan^2 A} \end{aligned} \quad (186)$$

Dividing num. and den. by $\tan A$:

$$= \frac{2}{\cot A - \tan A} \quad (187)$$

$$\begin{aligned} \cot 2A &= \cot (A + A) = \frac{\cot A \cot A - 1}{\cot A + \cot A} \\ &= \frac{\cot^2 A - 1}{2 \cot A} \end{aligned} \quad (188)$$

Dividing num. and den. by $\cot A$:

$$= \frac{\cot A - \tan A}{2} \quad (189)$$

$$\begin{aligned} \sec 2A &= \sec (A + A) = \frac{\sec A \sec A}{1 - \tan A \tan A}, \text{ by (163)} \\ &= \frac{\sec^2 A}{1 - \tan^2 A} \end{aligned} \quad (190)$$

and by the equation [h], page 15

$$= \frac{1 + \tan^2 A}{1 - \tan^2 A} \quad (191)$$

Also, by (164)

(185)

$$= \frac{\sec^2 A \operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - \sec^2 A} \quad (192)$$

(186)

$$\begin{aligned} \operatorname{cosec} 2 A &= \sin 2 A = \frac{1}{2 \sin A \cos A} \\ &= \frac{1}{2 \sin^2 A \cdot \frac{\cos A}{\sin A}}, \text{ which is evident.} \end{aligned}$$

(187)

Dividing num. and den. by $\sin^2 A$:

$$= \frac{\operatorname{cosec}^2 A}{2 \cot A} = \frac{1 + \cot^2 A}{2 \cot A} \quad (193)$$

Also, by (168):

(188)

$$= \frac{\sec^2 A \operatorname{cosec}^2 A}{2 \sec A \operatorname{cosec} A} \quad (194)$$

Dividing num. and den. by $\sec A \operatorname{cosec} A$:

(189)

$$= \frac{\sec A \operatorname{cosec} A}{2} \quad (195)$$

Angular Functions of 3 A .

163)

(190)

$$\begin{aligned} \sin 3 A &= \sin (2 A + A) = \sin 2 A \cos A + \cos 2 A \sin A \\ &= 2 \sin A \cos^2 A + (1 - 2 \sin^2 A) \sin A \\ &= 2 \sin (1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A \\ &= 3 \sin A - 4 \sin^3 A \end{aligned} \quad (196)$$

(191)

$$\begin{aligned} \cos 3 A &= \cos (2 A + A) = \cos 2 A \cos A - \sin 2 A \sin A \\ &= (2 \cos^2 A - 1) \cos A - 2 \cos A \sin^2 A \\ &= (2 \cos^2 A - 1) \cos A - 2 \cos A (1 - \cos^2 A) \end{aligned}$$

$$= 4 \cos^3 A - 3 \cos A \quad (197)$$

$$\tan 3 A = \tan (2 A + A) = \frac{\tan 2 A + \tan A}{1 - \tan 2 A \tan A}$$

$$= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} \cdot \tan A}, \text{ by (186)}$$

Multiplying num. and den. by $1 - \tan^2 A$, and reducing:

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \quad (198)$$

Dividing num. and den. by $\tan A$:

$$= \frac{3 - \tan^2 A}{\cot A - 3 \tan A} \quad (199)$$

$$\cot 3 A = \cot (2 A + A) = \frac{\cot 2 A \cot A - 1}{\cot 2 A + \cot A}$$

$$= \frac{\frac{\cot^2 A - 1}{2 \cot A} \cdot \cot A - 1}{\frac{\cot^2 A - 1}{2 \cot A} + \cot A}, \text{ by (187)}$$

Multiplying num. and den. by $2 \cot A$, and reducing:

$$= \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1} \quad (200)$$

Dividing num. and den. by $\cot A$.

$$(197) \quad = \frac{\cot^2 A - 3}{3 \cot A - \tan A} \quad (201)$$

$$\sin 4 A = \sin (3 A + A) = \sin 3 A \cos A + \cos 3 A \sin A$$

Substituting for $\sin 3 A$ and $\cos 3 A$ their values as found in (196) and (197), and reducing:

$$= 8 \sin A \cos^3 A - 4 \sin A \cos A \quad (202)$$

$$= (4 \sin A - 8 \sin^3 A) \sqrt{1 - \sin^2 A} \quad (203)$$

A , and re-

Following a similar process gives

$$\cos 4 A = 8 \cos^5 A - 8 \cos^3 A + 1 \quad (204)$$

$$= 1 - 8 \sin^2 A + 8 \sin^4 A \quad (205)$$

(198)

$$\tan 4 A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} \quad (206)$$

Dividing num. and den. by $\tan A$:

(199)

$$= \frac{4 - 4 \tan^2 A}{\cot A - 6 \tan A + \tan^3 A} \quad (207)$$

$$\cot 4 A = \frac{\cot^4 A - 6 \cot^2 A + 1}{4 \cot^3 A + 5 \cot A} \quad (208)$$

Dividing num. and den. by $\cot A$:

$$= \frac{\cot^3 A - 6 \cot A + \tan A}{4 \cot^2 A + 5} \quad (209)$$

and reducing:

(200)

By following a similar process of successive substitutions, we easily obtain the angular functions of 5, 6, 7... times A .

But we may more briefly obtain the proper result for any multiple Angle, by means of the following

General Formulae for Multiple Angles.

Substituting $(m-1) A$ for one of the angles in (155) and (156), we obtain

$$\begin{aligned}\sin [(m-1) A + A] &= \sin (m-1) A \cos A + \cos (m-1) A \sin A \\ \sin [(m-1) A - A] &= \sin (m-1) A \cos A - \cos (m-1) A \sin A\end{aligned}$$

Adding:

$$\sin [(m-1) A + A] + \sin [(m-1) A - A] = 2 \sin (m-1) A \cos A$$

Whence:

$$\sin [(m-1) A + A] = 2 \sin (m-1) A \cos A - \sin [(m-1) A - A]$$

or

$$\sin m A = 2 \sin (m-1) A \cos A - \sin (m-2) A \quad (210)$$

Proceeding in a similar way with (157) and (158), we obtain

$$\cos m A = 2 \cos (m-1) A \cos A - \cos (m-2) A \quad (211)$$

Substituting $(m-1) A$ for one of the angles in (159) and (161) gives

$$\tan m A = \frac{\tan (m-1) A + \tan A}{1 - \tan (m-1) A \tan A} \quad (212)$$

$$\cot m A = \frac{\cot (m-1) A \cot A - 1}{\cot (m-1) A + \cot A} \quad (213)$$

The formulæ (210), (211), (212), and (213), are general and may be applied to the computation of any multiple angle, by making m equal to 1, 2, 3, 4, 5, etc., and performing successive substitutions, as seen before. Thus, making $m = 2, 3, 4$, successively, we have by (210):

$$\begin{aligned}\sin 2 A &= 2 \sin (2-1) A \cos A - \sin (2-2) A \\ &= 2 \sin A \cos A = (179)\end{aligned}$$

$$\sin 3 A = 2 \sin (3-1) A \cos A - \sin (3-2) A$$

$$= 2 \sin 2 A \cos A - \sin A$$

which being reduced by substituting for $\sin 2 A$ its value ((179), gives (196).

$$\sin 4 A = 2 \sin (4 - 1) A \cos A - \sin (4 - 2) A$$

$$= 2 \sin 3 A \cos A - \sin 2 A$$

which being reduced by the successive substitution of $3 A$ and of $2 A$, gives (202).

And so on for the sine of any higher multiple of A , and also for the other functions of any multiple of A .

Angular Functions of $\frac{1}{2} A$.

Substituting $\frac{1}{2} A$ for A in (184) gives

$$\cos A = 1 - 2 \sin^2 \frac{1}{2} A, \text{ whence}$$

$$2 \sin^2 \frac{1}{2} A = 1 - \cos A; \text{ therefore,}$$

$$\sin \frac{1}{2} A = \sqrt{\left(\frac{1 - \cos A}{2}\right)} \quad (214)$$

$$= \sqrt{\left(\frac{1 - \sqrt{[1 - \sin^2 A]}}{2}\right)} \quad (215)$$

Substituting $\frac{1}{2} A$ for A in (183) gives

$$\cos A = 2 \cos^2 \frac{1}{2} A - 1; \text{ whence}$$

$$2 \cos^2 \frac{1}{2} A = 1 + \cos A; \text{ therefore}$$

$$\cos \frac{1}{2} A = \sqrt{\left(\frac{1 + \cos A}{2}\right)} \quad (216)$$

$$= \sqrt{\left(\frac{1 + \sqrt{[1 - \sin^2 A]}}{2}\right)} \quad (217)$$

$$\tan \frac{1}{2} A = \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}$$

$$= \sqrt{\left(\frac{1 - \cos A}{1 + \cos A}\right)} \quad (218)$$

Multiplying num. and den. by $\sqrt{1 - \cos A}$ or $\sqrt{1 + \cos A}$, and reducing :

$$= \frac{1 - \cos A}{\sin A} \quad (219)$$

$$= \frac{\sin A}{1 + \cos A} \quad (220)$$

$$\text{also} = \frac{\sin A}{\cos A} \div \left(\frac{1}{\cos A} + 1 \right) = \frac{\tan A}{\sec A + 1} \quad (221)$$

$$\begin{aligned} \cot \frac{1}{2} A &= \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A} \\ &= \sqrt{\left(\frac{1 + \cos A}{1 - \cos A} \right)} \end{aligned} \quad (222)$$

Proceeding as before :

$$= \frac{1 + \cos A}{\sin A} \quad (223)$$

$$= \frac{\sin A}{1 - \cos A} \quad (224)$$

$$= \frac{\sin A}{\cos A} \div \left(\frac{1}{\cos A} - 1 \right) = \frac{\tan A}{\sec A - 1} \quad (225)$$

$$\begin{aligned} \sec \frac{1}{2} A &= \frac{1}{\cos \frac{1}{2} A} = 1 \div \sqrt{\left(\frac{1 + \cos A}{2} \right)} \\ &= 1 \div \sqrt{\left(\frac{1}{2} + \frac{1}{2 \sec A} \right)} = 1 \div \sqrt{\left(\frac{1 + \sec A}{2 \sec A} \right)} \\ &= \sqrt{\left(\frac{2 \sec A}{1 + \sec A} \right)} \end{aligned} \quad (226)$$

$1 - \cos A]$ or

(219)

(220)

(221)

$$\operatorname{cosec} \frac{1}{2} A = \frac{1}{\sin \frac{1}{2} A} = 1 \div \sqrt{\left(\frac{1 - \cos A}{2}\right)}$$

$$= 1 \div \sqrt{\left(\frac{\sec A - 1}{2 \sec A}\right)}$$

$$= \sqrt{\left(\frac{2 \sec A}{\sec A - 1}\right)} \quad (227)$$

Formulas expressing the sum or the difference of two Functions including one or two variables.

By a simple algebraic artifice, we may take

$$A = \frac{1}{2} (A + B) + \frac{1}{2} (A - B)$$

$$B = \frac{1}{2} (A + B) - \frac{1}{2} (A - B); \text{ whence}$$

(222)

$$\sin A + \sin B = \sin \left[\frac{1}{2} (A + B) + \frac{1}{2} (A - B) \right] + \sin \left[\frac{1}{2} (A + B) - \frac{1}{2} (A - B) \right]$$

$$= \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$$

(223)

$$+ \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) - \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$$

(224)

$$= 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) \quad (228)$$

$$\sin A - \sin B = \sin \left[\frac{1}{2} (A + B) + \frac{1}{2} (A - B) \right] - \sin \left[\frac{1}{2} (A + B) - \frac{1}{2} (A - B) \right]$$

(225)

$$= \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$$

$$- \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$$

$$= 2 \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B) \quad (229)$$

$$\cos A + \cos B = \cos \left[\frac{1}{2} (A + B) + \frac{1}{2} (A - B) \right] + \cos \left[\frac{1}{2} (A + B) - \frac{1}{2} (A - B) \right]$$

$$= \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) - \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$$

$$+ \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B)$$

(226)

$$\sqrt{\left(\frac{1 + \sec A}{2 \sec A}\right)}$$

$$\begin{aligned} & \sin \frac{1}{2} (A - B) \\ &= 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) \end{aligned} \quad (230)$$

$$\begin{aligned} \cos A - \cos B &= \cos \left[\frac{1}{2} (A + B) + \frac{1}{2} (A - B) \right] - \cos \left[\frac{1}{2} (A + B) - \frac{1}{2} (A - B) \right] \\ &= \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) - \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B) \\ &= -\cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) - \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B) \\ &= -2 \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B) \end{aligned} \quad (231)$$

$$\begin{aligned} \sin A + \cos A &= \pm \sqrt{[\sin A + \cos A]^2}, \text{ which is evident} \\ &= \pm \sqrt{[\sin^2 A + \cos^2 A + 2 \sin A \cos A]} \\ &= \pm \sqrt{[1 + \sin 2A]}, \text{ by (179) and the equation [a]} \end{aligned} \quad (232)$$

that is $= +\sqrt{[1 + \sin 2A]}$

1. for all values of A whose sin and cos are both positive;
2. for values of A whose sin and cos have unlike signs, when sin is greater and positive, or less and negative. Otherwise:

$$= -\sqrt{[1 + \sin 2A]}$$

$$\begin{aligned} \sin A - \cos A &= \pm \sqrt{[\sin A - \cos A]^2}, \text{ which is evident} \\ &= \pm \sqrt{[\sin^2 A + \cos^2 A - 2 \sin A \cos A]} \\ &= \pm \sqrt{[1 - \sin 2A]} \end{aligned} \quad (233)$$

that is $= +\sqrt{[1 - \sin 2A]}$

1. for all values of A whose sin is positive and cos negative;
2. for values of A whose sin and cos have unlike signs when sin is greater and positive, or less and negative. Otherwise:

$$= -\sqrt{[1 - \sin 2A]}$$

$$\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$$

(230)

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}$$

$\cos \frac{1}{2} (A + B)$

$$= \frac{\sin (A + B)}{\cos A \cos B} \quad (234)$$

$\sin \frac{1}{2} (A + B)$

$\sin \frac{1}{2} (A + B)$

$$\text{also} = \frac{1}{\cot A} + \frac{1}{\cot B} = \frac{\cot B + \cot A}{\cot A \cot B} \quad (235)$$

(231)

is evident

$\cos A$

the equation [a]

(232)

and $\cos$ are both

$\cos$ have unlike
and positive, or

$$\tan A - \tan B = \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}$$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}$$

$$= \frac{\sin (A - B)}{\cos A \cos B} \quad (236)$$

$$\text{also} = \frac{1}{\cot A} - \frac{1}{\cot B} = \frac{\cot B - \cot A}{\cot A \cot B} \quad (237)$$

is evident

$\cos A$

(233)

in is positive and

and $\cos$ have un-
der and positive,
wise:

$$\cot A + \cot B = \frac{\cot A}{\sin A} + \frac{\cot B}{\sin B}$$

$$= \frac{\cos A \sin B + \sin A \cos B}{\sin A \sin B}$$

$$= \frac{\sin (A + B)}{\sin A \sin B} \quad (238)$$

$$\text{also} = \frac{1}{\tan A} + \frac{1}{\tan B} = \frac{\tan A + \tan B}{\tan A \tan B} \quad (239)$$

$$\cot A - \cot B = \frac{\cos A}{\sin B} - \frac{\cos B}{\sin A}$$

$$\begin{aligned}
 &= \frac{\cos A \sin B - \sin A \cos B}{\sin A \sin B} \\
 &= \frac{\sin (B - A)}{\sin A \sin B} \quad (240)
 \end{aligned}$$

$$\text{also} = \frac{1}{\tan A} - \frac{1}{\tan B} = \frac{\tan B - \tan A}{\tan B \tan A} \quad (241)$$

$$\begin{aligned}
 \tan A + \cot B &= \frac{\sin A}{\cos A} + \frac{\cos B}{\sin B} \\
 &= \frac{\cos A \cos B + \sin A \sin B}{\cos A \sin B} \\
 &= \frac{\cos (A - B)}{\cos A \sin B} \quad (242)
 \end{aligned}$$

$$\text{also} = \frac{1}{\cot A} + \frac{1}{\tan B} = \frac{\tan B + \cot A}{\tan B \cot A} \quad (243)$$

$$\begin{aligned}
 \cot A - \tan B &= \frac{\cos A}{\sin A} - \frac{\sin B}{\cos B} \\
 &= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B} \\
 &= \frac{\cos (A + B)}{\sin A \cos B} \quad (244)
 \end{aligned}$$

$$\text{also} = \frac{1}{\tan A} - \frac{1}{\cot B} = \frac{\cot B - \tan A}{\cot B \tan A} \quad (245)$$

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$\begin{aligned} &= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{1}{\sin A \cos A} \\ (240) \quad &= \frac{2}{2 \sin A \cos A} = \frac{2}{\sin 2 A} \quad (246) \end{aligned}$$

$$\begin{aligned} (241) \quad \cot A - \tan A &= \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} \\ &= \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} \\ &= \frac{\cos 2 A}{\sin A \cos A} = \frac{2 \cos 2 A}{\sin 2 A} \\ (242) \quad &= 2 \cot 2 A \quad (247) \end{aligned}$$

$$\begin{aligned} (243) \quad \sec A + \sec B &= \frac{1}{\cos A} + \frac{1}{\cos B} = \frac{\cos A + \cos B}{\cos A \cos B} \\ &= \frac{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\cos A \cos B} \quad (248) \end{aligned}$$

$$\begin{aligned} (244) \quad \sec B - \sec A &= \frac{1}{\cos B} - \frac{1}{\cos A} = \frac{\cos A - \cos B}{\cos A \cos B} \\ &= -\frac{2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{\cos A \cos B} \quad (249) \end{aligned}$$

$$\begin{aligned} (245) \quad \operatorname{cosec} A + \operatorname{cosec} B &= \frac{1}{\sin A} + \frac{1}{\sin B} = \frac{\sin A + \sin B}{\sin A \sin B} \\ &= \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\sin A \sin B} \quad (250) \end{aligned}$$

$$\begin{aligned}\operatorname{cosec} B + \operatorname{cosec} A &= \frac{1}{\sin B} + \frac{1}{\sin A} = \frac{\sin A + \sin B}{\sin A \sin B} \\ &= \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{\sin A \sin B} \quad (251)\end{aligned}$$

$$\sin(A+B) + \sin(A-B) = (155) + (156) = 2 \sin A \cos B \quad (252)$$

$$\sin(A+B) - \sin(A-B) = (155) - (156) = 2 \cos A \sin B \quad (253)$$

$$\cos(A+B) + \cos(A-B) = (157) + (158) = 2 \cos A \cos B \quad (254)$$

$$\cos(A+B) - \cos(A-B) = (157) - (158) = 2 \sin A \sin B \quad (255)$$

$$\cos(A-B) - \cos(A+B) = (158) - (157) = 2 \sin A \sin B \quad (256)$$

$$\begin{aligned}\tan(A+B) + \tan(A-B) &= (159) + (160) \\ &= \frac{2 \tan A + 2 \tan A \tan^2 B}{1 - \tan^2 A \tan^2 B} \quad (257)\end{aligned}$$

$$\begin{aligned}\tan(A+B) - \tan(A-B) &= (159) - (160) \\ &= \frac{2 \tan B + 2 \tan^2 A \tan B}{1 - \tan^2 A \tan^2 B} \quad (258)\end{aligned}$$

$$\begin{aligned}\cot(A+B) + \cot(A-B) &= (161) + (162) \\ &= \frac{2 \cot A + 2 \cot A \cot^2 B}{\cot^2 B - \cot^2 A} \quad (259)\end{aligned}$$

$\cot(A+B)$

$\sec(A+B)$

$\sec(A+B)$

$\operatorname{cosec}(A+B)$

$\operatorname{cosec}(A+B)$

Let m be
vious that

(228), (229),
 $\sin(m+B)$

$$\cot(A+B) - \cot(A-B) = (161) + (162)$$

$$= \frac{-2 \cot B - 2 \cot^2 A \cot B}{\cot^2 B - \cot^2 A} \quad (260)$$

$$\sec(A+B) + \sec(A-B) = (164) + (166)$$

$$= \frac{2 \sec A \sec B \operatorname{cosec}^2 A \operatorname{cosec}^2 B}{\operatorname{cosec}^2 A \operatorname{cosec}^2 B - \sec^2 A \sec^2 B} \quad (261)$$

$$\sec(A+B) - \sec(A-B) = (164) - (166)$$

$$= \frac{2 \sec^2 A \sec^2 B \operatorname{cosec} A \operatorname{cosec} B}{\operatorname{cosec}^2 A \operatorname{cosec}^2 B - \sec^2 A \sec^2 B} \quad (262)$$

$$\operatorname{cosec}(A+B) + \operatorname{cosec}(A-B) = (168) + (170)$$

$$= \frac{2 \sec^2 A \sec B \operatorname{cosec} A \operatorname{cosec}^2 B}{\sec^2 A \operatorname{cosec}^2 B - \sec^2 B \operatorname{cosec}^2 A} \quad (263)$$

$$\operatorname{cosec}(A+B) - \operatorname{cosec}(A-B) = (168) - (170)$$

$$= \frac{2 \sec A \operatorname{cosec} B \operatorname{cosec}^2 A \sec^2 B}{\sec^2 A \operatorname{cosec}^2 B - \operatorname{cosec}^2 A \sec^2 B} \quad (264)$$

Let m be any positive number greater than unity; then it is obvious that

$$\frac{1}{2}(m+1) + \frac{1}{2}(m-1) = m$$

$$\frac{1}{2}(m+1) - \frac{1}{2}(m-1) = 1$$

$$\text{or } (m+1) + (m-1) = 2m$$

$$(m+1) - (m-1) = 2; \text{ whence, by}$$

(228), (229), (230), (231):

$$\begin{aligned} \sin(m+1)A + \sin(m-1)A &= 2 \sin\left[\frac{1}{2}(m+1)A + \frac{1}{2}(m-1)A\right] \\ &\quad \times \cos\left[\frac{1}{2}(m+1)A - \frac{1}{2}(m-1)A\right] \\ &= 2 \sin mA \cos A \end{aligned} \quad (265)$$

$$\begin{aligned}\sin (m+1) A - \sin (m-1) A &= 2 \cos \left[\frac{1}{2} (m+1) A + \frac{1}{2} (m-1) A \right] \\ &\times \sin \left[\frac{1}{2} (m+1) A - \frac{1}{2} (m-1) A \right] \\ &= 2 \cos m A \sin A \quad (266)\end{aligned}$$

$$\begin{aligned}\cos (m+1) A + \cos (m-1) A &= 2 \cos \left[\frac{1}{2} (m+1) A + \frac{1}{2} (m-1) A \right] \\ &\times \cos \left[\frac{1}{2} (m+1) A - \frac{1}{2} (m-1) A \right] \\ &= 2 \cos m A \cos A \quad (267)\end{aligned}$$

$$\begin{aligned}\cos (m+1) A - \cos (m-1) A &= -2 \sin \left[\frac{1}{2} (m+1) A + \frac{1}{2} (m-1) A \right] \\ &\times \sin \left[\frac{1}{2} (m+1) A - \frac{1}{2} (m-1) A \right] \\ &= -2 \sin m A \sin A \quad (268)\end{aligned}$$

$$\begin{aligned}\tan (m+1) A + \tan (m-1) A &= \frac{\sin [(m+1) A + (m-1) A]}{\cos (m+1) A \cos (m-1) A}, \\ &\text{by (234)} \\ &= \frac{\sin 2 m A}{\cos (m+1) A \cos (m-1) A} \quad (269)\end{aligned}$$

$$\begin{aligned}\tan (m+1) A - \tan (m-1) A &= \frac{\sin [(m+1) A - (m-1) A]}{\cos (m+1) A \cos (m-1) A}, \\ &\text{by (236)} \\ &= \frac{\sin 2 A}{\cos (m+1) A \cos (m-1) A} \quad (270)\end{aligned}$$

$$\begin{aligned}\cot (m+1) A + \cot (m-1) A &= \frac{\sin [(m+1) A + (m-1) A]}{\sin (m+1) A \sin (m-1) A}, \\ &\text{by (238)} \\ &= \frac{\sin 2 m A}{\sin (m+1) A \sin (m-1) A} \quad (271)\end{aligned}$$

$$\frac{1}{2}(m-1)A] \\ - \frac{1}{2}(m-1)A] \\ (266)$$

$$\cot(m-1)A - \cot(m+1)A = \frac{\sin[(m+1)A - (m-1)A]}{\sin(m+1)A \sin(m-1)A}, \\ \text{by (240)}$$

$$\frac{1}{2}(m-1)A] \\ - \frac{1}{2}(m-1)A] \\ (267)$$

$$= \frac{\sin 2A}{\sin(m+1)A \sin(m-1)A} \quad (272)$$

$$\tan(m+1)A + \cot(m-1)A = \frac{\cos[(m+1)A - (m-1)A]}{\cos(m+1)A \sin(m-1)A}, \\ \text{by (242)}$$

$$\frac{1}{2}(m-1)A] \\ - \frac{1}{2}(m-1)A] \\ (268)$$

$$= \frac{\cos 2A}{\cos(m+1)A \sin(m-1)A} \quad (273)$$

$$\frac{(m-1)A]}{\cos(m-1)A},$$

$$\cot(m+1)A - \tan(m-1)A = \frac{\cos[(m+1)A + (m-1)A]}{\sin(m+1)A \cos(m-1)A}, \\ \text{by (244)}$$

$$\frac{(m-1)A]}{\cos(m-1)A},$$

$$= \frac{\cos 2mA}{\sin(m+1)A \cos(m-1)A} \quad (274)$$

$$\frac{(m-1)A]}{\cos(m-1)A},$$

By applying the same method of substitution and reduction to the formulæ (248), (219), (250), (251), we easily obtain:

$$\sec(m+1)A + \sec(m-1)A = \frac{2 \cos mA \cos A}{\cos(m+1)A \cos(m-1)A} \quad (275)$$

$$\frac{(m-1)A]}{\cos(m-1)A},$$

$$\sec(m-1)A - \sec(m+1)A = -\frac{2 \sin mA \sin A}{\cos(m+1)A \cos(m-1)A} \quad (276)$$

$$\frac{(m-1)A]}{\sin(m-1)A},$$

$$\operatorname{cosec}(m+1)A + \operatorname{cosec}(m-1)A = \frac{2 \sin mA \cos A}{\sin(m+1)A \sin(m-1)A} \quad (277)$$

$$\frac{(m-1)A]}{\sin(m-1)A},$$

$$\operatorname{cosec}(m-1)A - \operatorname{cosec}(m+1)A = \frac{2 \cos mA \sin A}{\sin(m+1)A \sin(m-1)A} \quad (278)$$

Let A be 45° , 30° , or 60° , in (252), (253), (254), and (255); then we obtain immediately :

$$\begin{aligned}\sin (45^\circ + B) + \sin (45^\circ - B) &= 2 \sin 45^\circ \cos B \\ &= \cos B \sqrt{2}\end{aligned}\quad (279)$$

$$\begin{aligned}\sin (45^\circ + B) - \sin (45^\circ - B) &= 2 \cos 45^\circ \sin B \\ &= \sin B \sqrt{2}\end{aligned}\quad (280)$$

$$\begin{aligned}\cos (45^\circ + B) + \cos (45^\circ - B) &= 2 \cos 45^\circ \cos B \\ &= \cos B \sqrt{2}\end{aligned}\quad (281)$$

$$\begin{aligned}\cos (45^\circ + B) - \cos (45^\circ - B) &= -2 \sin 45^\circ \sin B \\ &= -2 \sin B \sqrt{2}\end{aligned}\quad (282)$$

$$\begin{aligned}\sin (30^\circ + B) + \sin (30^\circ - B) &= 2 \sin 30^\circ \cos B \\ &= \cos B\end{aligned}\quad (283)$$

$$\begin{aligned}\sin (30^\circ + B) - \sin (30^\circ - B) &= 2 \cos 30^\circ \sin B \\ &= \sin B \sqrt{3}\end{aligned}\quad (284)$$

$$\begin{aligned}\cos (30^\circ + B) + \cos (30^\circ - B) &= 2 \cos 30^\circ \cos B \\ &= \cos B \sqrt{3}\end{aligned}\quad (285)$$

$$\begin{aligned}\cos (30^\circ + B) - \cos (30^\circ - B) &= -2 \sin 30^\circ \sin B \\ &= -\sin B\end{aligned}\quad (286)$$

$$\begin{aligned}\sin (60^\circ + B) + \sin (60^\circ - B) &= 2 \sin 60^\circ \cos B \\ &= \cos B \sqrt{3}\end{aligned}\quad (287)$$

$$\begin{aligned}\sin (60^\circ + B) - \sin (60^\circ - B) &= 2 \cos 60^\circ \sin B \\ &= \sin B\end{aligned}\quad (288)$$

$$\begin{aligned}\cos (60^\circ + B) + \cos (60^\circ - B) &= 2 \cos 60^\circ \cos B \\ &= \cos B\end{aligned}\quad (289)$$

$$\begin{aligned}\cos (60^\circ + B) - \cos (60^\circ - B) &= -2 \sin 60^\circ \sin B \\ &= -\sin B \sqrt{3}\end{aligned}\quad (290)$$

Formu

$\sin (A +$

$\cos (A +$

$\tan (A +$

*Formulae expressing the Product of two Functions including,
each, one or two Variables.*

(255); then

(279)

(280)

(281)

(282)

(283)

(284)

(285)

(286)

(287)

(288)

(289)

(290)

$$\begin{aligned}\sin (A+B) \sin (A-B) &= (155) \times (156) \\ &= (\sin A \cos B + \cos A \sin B) \\ &\quad \times (\sin A \cos B - \cos A \sin B) \\ &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ &= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \\ &\quad \times \sin^2 B \\ &= \sin^2 A - \sin^2 B\end{aligned}\quad (291)$$

$$\begin{aligned}\text{or} &= (1 - \cos^2 A) \cos^2 B - \cos^2 A \\ &\quad \times (1 - \cos^2 B) \\ &= \cos^2 B - \cos^2 A\end{aligned}\quad (292)$$

$$\begin{aligned}\cos (A+B) \cos (A-B) &= (157) \times (158) \\ &= (\cos A \cos B - \sin A \sin B) \\ &\quad \times (\cos A \cos B + \sin A \sin B) \\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ &= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \\ &\quad \times \sin^2 B \\ &= \cos^2 A - \sin^2 B\end{aligned}\quad (293)$$

$$\text{or} = \cos^2 B - \sin^2 A\quad (294)$$

$$\text{or} = \cos^2 A + \cos^2 B - 1\quad (295)$$

$$\begin{aligned}\tan (A+B) \tan (A-B) &= \frac{\sin (A+B) \sin (A-B)}{\cos (A+B) \cos (A-B)} \\ &= \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}\end{aligned}\quad (296)$$

$$= \frac{\cos^2 B - \cos^2 A}{\cos^2 B - \sin^2 A}\quad (297)$$

$$\text{also} = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}\quad (298)$$

$$\begin{aligned}\cot(A+B) \cot(A-B) &= \frac{\cos(A+B) \cos(A-B)}{\sin(A+B) \sin(A-B)} \\ &= \frac{\cos^2 A - \sin^2 B}{\sin^2 A - \sin^2 B} \quad (299)\end{aligned}$$

$$= \frac{\cos^2 B - \sin^2 A}{\cos^2 B - \cos^2 A} \quad (300)$$

$$\text{also} = (161) \times (162) = \frac{\cot^2 A \cot^2 B - 1}{\cot^2 B - \cot^2 A} \quad (301)$$

$$\begin{aligned}\sec(A+B) \sec(A-B) &= (164) \times (166) \\ &= \frac{\sec^2 A \sec^2 B - \operatorname{cosec}^2 A \operatorname{cosec}^2 B}{\operatorname{cosec}^2 A \operatorname{cosec}^2 B - \sec^2 A \sec^2 B} \quad (302)\end{aligned}$$

$$\operatorname{cosec}(A+B) \operatorname{cosec}(A-B) = \frac{\sec^2 A \sec^2 B \operatorname{cosec}^2 A \operatorname{cosec}^2 B}{\sec^2 A \operatorname{cosec}^2 B - \sec^2 B \operatorname{cosec}^2 A} \quad (303)$$

In the preceding formulæ, write $\frac{1}{2}(A+B)$, for A ; and $\frac{1}{2}(A-B)$, for B , then since $\frac{1}{2}(A+B) + \frac{1}{2}(A-B) = A$ and $\frac{1}{2}(A+B) - \frac{1}{2}(A-B) = B$, we obtain

$$\begin{aligned}\sin A \sin B &= \sin^2 \frac{1}{2}(A+B) - \sin^2 \frac{1}{2}(A-B) \quad (304) \\ \text{or} &= \cos^2 \frac{1}{2}(A-B) - \cos^2 \frac{1}{2}(A+B) \quad (305)\end{aligned}$$

$$\begin{aligned}\cos A \cos B &= \cos^2 \frac{1}{2}(A+B) - \sin^2 \frac{1}{2}(A-B) \quad (306) \\ \text{or} &= \cos^2 \frac{1}{2}(A-B) - \sin^2 \frac{1}{2}(A+B) \quad (307)\end{aligned}$$

$$\tan A \tan B = \frac{\sin^2 \frac{1}{2}(A+B) - \sin^2 \frac{1}{2}(A-B)}{\cos^2 \frac{1}{2}(A+B) - \sin^2 \frac{1}{2}(A-B)} \quad (308)$$

$$\text{or} = \frac{\cos^2 \frac{1}{2}(A-B) - \cos^2 \frac{1}{2}(A+B)}{\cos^2 \frac{1}{2}(A-B) - \sin^2 \frac{1}{2}(A-B)} \quad (309)$$

$\cot A \cot B$

Formula

$\frac{\sin(A+B)}{\sin(A-B)}$

Dividing
ducing

$$\cot A \cot B = \frac{\cos^2 \frac{1}{2} (A+B) - \sin^2 \frac{1}{2} (A-B)}{\sin^2 \frac{1}{2} (A+B) - \sin^2 \frac{1}{2} (A-B)} \quad (310)$$

$$= \frac{\cos^2 \frac{1}{2} (A-B) - \sin^2 \frac{1}{2} (A+B)}{\cos^2 \frac{1}{2} (A-B) - \cos^2 \frac{1}{2} (A+B)} \quad (311)$$

(300)

*Formulas expressing the quotient of Functions including,
each, one or two Variables.*

$$\frac{\cot^2 B - 1}{\cot^2 A} \quad (301)$$

$$\frac{\sin (A+B)}{\sin (A-B)} = \frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B}$$

$$\frac{\operatorname{cosec}^2 B}{A \sec^2 B} \quad (302)$$

Dividing num. and den. by $\cos A \cos B$ or $\sin A \sin B$, and reducing

$$\frac{\operatorname{cosec}^2 B}{\operatorname{cosec}^2 A} \quad (303)$$

$$= \frac{\tan A + \tan B}{\tan A - \tan B} \quad (312)$$

$$\text{and } \frac{1}{2} (A-B),$$

$$= \frac{\cot B + \cot A}{\cot B - \cot A} \quad (313)$$

obtain

$$(304)$$

$$(305)$$

Also dividing num. and den. by $\sin A \cos B$, or $\cos A \sin B$ +.

$$(306)$$

$$(307)$$

$$= \frac{1 + \tan B \cot A}{1 - \tan B \cot A} \quad (314)$$

$$(308)$$

$$(309)$$

$$= \frac{\tan A \cot B + 1}{\tan A \cot B - 1} \quad (315)$$

$$\frac{\cos (A+B)}{\cos (A-B)} = \frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B + \sin A \sin B}$$

Dividing num. and den. by $\cos A \sin B$, or
 $\sin A \cos B$, and reducing:

$$= \frac{\cot B - \tan A}{\cot B + \tan A} \quad (316)$$

$$= \frac{\cot A - \tan B}{\cot A + \tan B} \quad (317)$$

Also dividing num. and den. by $\cos A \cos B$,
 or $\sin A \sin B$, and reducing:

$$= \frac{1 - \tan A \tan B}{1 + \tan A \tan B} \quad (318)$$

$$= \frac{\cot A \cot B - 1}{\cot A \cot B + 1} \quad (319)$$

By following a similar process, we obtain easily

$$\begin{aligned} \frac{\tan (A+B)}{\cos (A+B)} &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{1 + \cot A \tan B}{\cot A - \tan B} \quad (320) \end{aligned}$$

$$= \frac{\tan A \cot B + 1}{\cot B - \tan A} \quad (321)$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad (322)$$

$$= \frac{\cot B + \cot A}{\cot A \cot B - 1} \quad (323)$$

$$(316) \quad \frac{\sin(A - B)}{\cos(A - B)} = \frac{1 - \cot A \tan B}{\cot A + \tan B} \quad (324)$$

$$= \frac{\tan A \cot B - 1}{\cot B + \tan A} \quad (325)$$

$$(317) \quad = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad (326)$$

$$s A \cos B, \quad = \frac{\cot B - \cot A}{\cot A \cot B + 1} \quad (327)$$

$$(318) \quad \frac{\sin(A + B)}{\cos(A - B)} = \frac{1 + \cot A \tan B}{\cot A + \tan B} \quad (328)$$

$$(319) \quad = \frac{\tan A \cot B + 1}{\cot B + \tan A} \quad (329)$$

$$= \frac{\tan A + \tan B}{1 + \tan A \tan B} \quad (330)$$

$$= \frac{\cot B + \cot A}{\cot A \cot B + 1} \quad (331)$$

$$(320) \quad \frac{\sin(A + B)}{\sin A \cos B} = \frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B} \quad (332)$$

$$= 1 + \cot A \tan B$$

$$(321) \quad \frac{\sin(A - B)}{\sin A \cos B} = \frac{\sin A \cos B - \cos A \sin B}{\sin A \cos B}$$

$$= 1 - \cot A \tan B \quad (333)$$

$$\begin{aligned} \frac{\cos(A+B)}{\sin A \sin B} &= \frac{\cos A \cos B - \sin A \sin B}{\sin A \sin B} \\ &= \cot A \cot B - 1 \end{aligned} \quad (334)$$

$$\begin{aligned} \frac{\cos(A-B)}{\sin A \sin B} &= \frac{\cos A \cos B + \sin A \sin B}{\sin A \sin B} \\ &= \cot A \cot B + 1 \end{aligned} \quad (335)$$

$$\begin{aligned} \frac{\cos(A+B)}{\cos A \cos B} &= \frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B} \\ &= 1 - \tan A \tan B \end{aligned} \quad (336)$$

$$\begin{aligned} \frac{\cos(A-B)}{\cos A \cos B} &= \frac{\cos A \cos B + \sin A \sin B}{\cos A \cos B} \\ &= 1 + \tan A \tan B \end{aligned} \quad (337)$$

See also (234), (236), (238), (240), (242), and (244).

$$\begin{aligned} \frac{\sin A + \sin B}{\sin A - \sin B} &= \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)} \\ &= \tan \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B) \end{aligned} \quad (338)$$

$$= \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \quad (339)$$

$$\begin{aligned} \frac{\cos A + \cos B}{\cos A - \cos B} &= \frac{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{-2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)} \\ &= -\cot \frac{1}{2}(A+B) \cot \frac{1}{2}(A-B) \end{aligned} \quad (340)$$

$$\frac{\sin A + \sin B}{\cos A + \cos B}$$

$$\frac{\sin A - \sin B}{\cos A + \cos B}$$

$$\frac{\sin A + \sin B}{\cos A - \cos B}$$

$$\frac{\sin A - \sin B}{\cos A - \cos B}$$

$$\frac{\sin(A+B)}{\sin(A-B)}$$

$$\frac{\cos(A+B)}{\cos(A-B)}$$

$$\frac{\sin(A+B)}{\cos(A+B)}$$

$$\frac{\sin(A+B)}{\cos(A+B)}$$

$$\frac{\sin(A+B)}{\cos(A+B)}$$

$$\frac{\sin(A+B)}{\cos(A+B)}$$

By (183)

$$(333) \quad = -\frac{\cot \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \quad (341)$$

$$(334) \quad \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{(228)}{(230)} = \tan \frac{1}{2}(A+B) \quad (342)$$

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{(229)}{(230)} = \tan \frac{1}{2}(A-B) \quad (343)$$

$$(335) \quad \frac{\sin A + \sin B}{\cos A - \cos B} = \frac{(228)}{(231)} = -\cot \frac{1}{2}(A-B) \quad (344)$$

$$\frac{\sin A - \sin B}{\cos A - \cos B} = \frac{(229)}{(231)} = -\cot \frac{1}{2}(A+B) \quad (345)$$

$$(336) \quad \frac{\sin(A+B) + \sin(A-B)}{\sin(A+B) - \sin(A-B)} = \frac{(252)}{(253)} = \tan A \cot B \quad (346)$$

$$(337) \quad \frac{\cos(A+B) + \cos(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{(254)}{(255)} = -\cot A \cot B \quad (347)$$

$$\frac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \frac{(252)}{(254)} = \tan A \quad (348)$$

$$\frac{\sin(A+B) + \sin(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{(252)}{(255)} = \cot B \quad (349)$$

$$(338) \quad \frac{\sin(A+B) - \sin(A-B)}{\cos(A+B) - \cos(A-B)} = \frac{(253)}{(255)} = \cot A \quad (350)$$

(339)

Deductions.

By (183) and (184), we have

$$(340) \quad 2 \cos^2 A = 1 + \cos 2A \quad (351)$$

$$2 \sin^2 A = 1 - \cos 2A \quad (352)$$

Hence

$$\sin^2 A = \frac{1 - \cos 2A}{2} \quad (353)$$

$$\cos^2 A = \frac{1 + \cos 2A}{2} \quad (354)$$

$$\sin A = \sqrt{\left(\frac{1 - \cos 2A}{2}\right)} \quad (355)$$

$$\cos A = \sqrt{\left(\frac{1 + \cos 2A}{2}\right)} \quad (356)$$

Also

$$2 \sin^2 \frac{1}{2} A = 1 - \cos A \quad (357)$$

$$2 \cos^2 \frac{1}{2} A = 1 + \cos A \quad (358)$$

$$\sin^2 \frac{1}{2} A = \frac{1 - \cos A}{2} \quad (359)$$

$$\cos^2 \frac{1}{2} A = \frac{1 + \cos A}{2} \quad (360)$$

$$\sin \frac{1}{2} A = \sqrt{\left(\frac{1 - \cos A}{2}\right)} \quad (361)$$

$$\cos \frac{1}{2} A = \sqrt{\left(\frac{1 + \cos A}{2}\right)} \quad (362)$$

$$2 \sin A = \sqrt{[2 - 2 \cos 2A]} \quad (363)$$

$$2 \cos A = \sqrt{[2 + 2 \cos A]} \quad (364)$$

$$2 \sin \frac{1}{2} A = \sqrt{[2 - 2 \cos A]} \quad (365)$$

$$2 \cos \frac{1}{2} A = \sqrt{[2 + 2 \cos A]} \quad (366)$$

Whence

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A} \quad (367)$$

In (179) su

Hence sin

whence sin

By (196)

By (179) an

$$(353) \quad \cot^2 A = \frac{1 + \cos 2A}{1 - \cos 2A} \quad (368)$$

$$(354) \quad \tan A = \sqrt{\left(\frac{1 - \cos 2A}{1 + \cos 2A}\right)} \quad (369)$$

$$(355) \quad \cot A = \sqrt{\left(\frac{1 + \cos 2A}{1 - \cos 2A}\right)} \quad (370)$$

$$(356) \quad \tan^2 \frac{1}{2} A = \frac{1 - \cos A}{1 + \cos A} \quad (371)$$

$$(357) \quad \cot^2 \frac{1}{2} A = \frac{1 + \cos A}{1 - \cos A} \quad (372)$$

(359) In (179) substituting $\frac{1}{2} A$ for A gives

$$(360) \quad \sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A \quad (373)$$

$$(361) \quad \text{Hence } \sin (A + B) = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A + B) \quad (374)$$

$$(362) \quad \text{whence } \frac{\sin A + \sin B}{\sin (A + B)} = \frac{\cos \frac{1}{2} (A - B)}{\cos \frac{1}{2} (A + B)} \quad (375)$$

$$(363) \quad \text{By (196)} \quad \sin A = 3 \sin \frac{1}{3} A - 4 \sin^3 \frac{1}{3} A$$

(364) By (179) and equation [a], page 13, we have

$$(365) \quad \sin 2A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$$

(366) Dividing num. and den. by $\cos^2 A$

(367)

$$= \frac{2 \tan A}{1 + \tan^2 A} \quad (376)$$

whence $\sin A = \frac{2 \tan \frac{1}{2} A}{1 + \tan^2 \frac{1}{2} A} \quad (377)$

Dividing num. and den. by $\tan \frac{1}{2} A$

$$= \frac{2}{\cot \frac{1}{2} A + \tan \frac{1}{2} A} \quad (378)$$

$$\tan^2 A - \tan^2 B = (\tan A + \tan B) (\tan A - \tan B)$$

$$= (234) \times (236)$$

$$= \frac{\sin (A+B) \sin (A-B)}{\cos^2 A \cos^2 B} \quad (379)$$

$$\cot^2 A - \cot^2 B = (\cot A + \cot B) (\cot A - \cot B)$$

$$= (238) \times (240)$$

$$= \frac{\sin (A+B) \sin (B-A)}{\sin^2 A \sin^2 B} \quad (380)$$

$$= \frac{-\sin (A+B) \sin (A-B)}{\sin^2 A \sin^2 B} \quad (381)$$

Functions involving three Angles.

Let A be any three angles; then $\sin (A+B+C)$ and $\cos (A+B+C)$ may easily be expanded by making use of the formulæ (155) and (157) thus,

$$\begin{aligned} \sin (A+B+C) &= \sin (A+B) \cos C + \cos (A+B) \sin C \\ &= \sin A \cos B \cos C + \sin B \cos A \cos C \\ &\quad + \sin C \cos A \cos B - \sin A \sin B \sin C \quad (382) \end{aligned}$$

It is manifest that the expansion of $\sin (A+B+C)$ may be obtained by making use of the formulæ (155) and (157) thus, We have $\sin (A+B) \cos C + \cos (A+B) \sin C = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$

$$\begin{aligned} \cos(A+B+C) &= \cos(A+B) \cos C - \sin(A+B) \sin C \\ &= \cos A \cos B \cos C - \cos A \sin B \sin C \\ &\quad - \cos B \sin A \sin C - \cos C \sin A \sin B \end{aligned} \quad (383)$$

$$\tan(A+B+C) = \frac{\sin(A+B+C)}{\cos(A+B+C)} \quad (377)$$

Expanding num. and den. and dividing both by $\cos A \cos B \cos C$,

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C} \quad (384)$$

$$\cot(A+B+C) = \frac{\cos(A+B+C)}{\sin(A+B+C)} \quad (379)$$

Expanding and dividing by $\sin A \sin B \sin C$,
or by $\cos A \cos B \cos C$:

$$= \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot A \cot C + \cot A \cot B - 1} \quad (385)$$

$$\text{or} = \frac{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}{\tan A + \tan B + \tan C - \tan A \tan B \tan C} \quad (386)$$

It is manifest that in the same way, expressions for the Functions of any compound Angle may be found in terms of the functions of the component angles.

We have also, by applying successively (252) and (155):

$$\begin{aligned} &\sin(A+B+C) + \sin(B+C-A) \sin(A+C-B) \\ &\quad + \sin(A+B-C) \\ &= \sin[(B+C)+A] + \sin[(B+C)-A] + \sin[A-(B-C)] \\ &\quad + \sin[A+(B-C)] \\ &= 2 \sin(B+C) \cos A + 2 \sin A \cos(B-C) \end{aligned} \quad (382)$$

$$= 2 \sin A \sin B \sin C + 2 \sin A \cos B \cos C \\ + 2 \sin B \cos A \cos C + 2 \sin C \cos A \cos B \quad (387)$$

By applying successively (254) (157):

$$\cos (A + B + C) + \cos (B + C - A) + \cos (A + C - B) \\ + \cos (A + B - C) \\ = \cos [(B + C) + A] + \cos [(B + C) - A] + \cos [A - (B - C)] \\ + \cos [A + (B - C)]$$

$$= 2 \cos (B + C) \cos A + 2 \cos (B - C) \cos A$$

$$= 2 \cos A \cos B \cos C$$

$$= 4 \cos A \cos B \cos C \quad (388)$$

$$= (A + B) \sin (B + C) = \sin (A + B) \sin B \cos C \\ + \sin (A + B) \cos B \sin C$$

$$= \sin (A + B) \sin B \cos C + \cos A \sin B \\ \cos B \sin C + \sin A \sin C \cos^2 B$$

Substituting for $\cos^2 B$ in the last term its equal $1 - \sin^2 B$, and factoring:

$$= \sin B [\sin (A + B) \cos C + \sin C \\ \times (\cos A \cos B - \sin A \sin B)] \\ + \sin A \sin C$$

$$= \sin A \sin C + \sin B [\sin (A + B) \\ \times \cos C + \sin C \cos (A + B)]$$

$$= \sin A \sin C + \sin B \sin (A + B + C) \quad (389)$$

The student will have no difficulty in deducing the following:

$$(387) \quad \frac{\sin(A+B+C)}{\cos A \cos B \cos C} = \tan A + \tan B + \tan C - \tan A \tan B \tan C$$

(390)

$$\frac{\sin(A+B+C)}{\sin A \sin B \sin C} = \cot A \cot B + \cot A \cot C + \cot B \cot C - 1$$

(391)

$$\frac{\cos(A+B+C)}{\sin A \sin B \sin C} = \cot A \cot B \cot C - \cot A - \cot B - \cot C$$

(392)

$$(388) \quad \frac{\cos(A+B+C)}{\cos A \cos B \cos C} = 1 - \tan A \tan B - \tan A \tan C - \tan B \tan C$$

(393)

$$\sin A \sin(B-C) + \sin B \sin(C-A) + \sin C \sin(A-B) =$$

(394)

$$\cos A \sin(B-C) + \cos B \sin(C-A) + \cos C \sin(A-B) = 0$$

(395)

$$\sin A + \sin B + \sin C = 4 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A+C) \sin \frac{1}{2}(B+C)$$

(396)

$$\cos A + \cos B + \cos C = 4 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A+C) \cos \frac{1}{2}(B+C)$$

(397)

$$\sin(A-B) + \sin(B-C) + \sin(C-A) = -4 \sin \frac{1}{2}(A-B) \sin \frac{1}{2}(B-C) \sin \frac{1}{2}(C-A)$$

(398)

$$\cos 2A + \cos 2B + \cos 2C + \cos 2(A+B+C) = 4 \cos(A+B) \cos(B+C) \cos(C+A)$$

(399)

the following:

Particular Case of $A + B + C = 180^\circ$

Let $A + B + C = 180^\circ$ or be the angles of any triangle, then

$$\sin (A + B + C) = \sin 180^\circ = 0 \quad (400)$$

$$\cos (A + B + C) = \cos 180^\circ = -1 \quad (401)$$

$$\sin A + \sin B + \sin C = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + \sin C$$

But $\sin C = 2 \sin \frac{1}{2} C \cos \frac{1}{2} C$,
 $\sin \frac{1}{2} C = \cos \frac{1}{2} (A + B)$, by (81)
 $\cos \frac{1}{2} C = \sin \frac{1}{2} (A + B)$, by (82)

Therefore

$$\begin{aligned} \sin A + \sin B + \sin C &= 2 \sin \frac{1}{2} (A + B) [\cos \frac{1}{2} (A - B) \\ &\quad + \cos (A + B)] \\ &= 2 \cos \frac{1}{2} C [2 \cos \frac{1}{2} A \cos \frac{1}{2} B \\ &= 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C \end{aligned} \quad (402)$$

$$\begin{aligned} \cos A + \cos B + \cos C &= 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + \cos C \\ &= 2 \sin \frac{1}{2} C \cos \frac{1}{2} (A - B) + 1 - 2 \sin^2 \frac{1}{2} C \\ &= 1 + 2 \sin \frac{1}{2} C [\cos \frac{1}{2} (A - B) - \sin^2 \frac{1}{2} C] \\ &= 1 + 2 \sin \frac{1}{2} C [\cos \frac{1}{2} (A - B) \\ &\quad - \cos (A + B)] \\ &= 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C \end{aligned} \quad (403)$$

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= 2 \sin (A + B) \cos (A - B) \\ &\quad + 2 \sin C \cos C \\ &= 2 \sin C \cos (A - B) - 2 \sin C \cos \\ &\quad (A + B) \\ &= 2 \sin C [\cos (A - B) - \cos (A + B)] \end{aligned}$$

$$\sin (A + B)$$

$$\sin (A + B)$$

Since $\sin^2 \frac{1}{2} A$

Therefore $\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C = 1$

Since $\cos^2 A$
 $= 1 + \sin^2 A$

$$= 1 +$$

$$= 1 +$$

$$= 1 -$$

$$= 4 \sin A \sin B \sin C \quad (404)$$

$$\sin(A+B) \sin(B+C) = \sin(180^\circ - C) \sin(180^\circ - A) \\ = \sin A \sin C \quad (405)$$

$$\sin(A+B+C) + \sin(A+B-C) + \sin(A+C-B) \\ + \sin(B+C-A) \\ = \sin 180^\circ + \sin(180^\circ - 2C) + \sin \\ (180^\circ - 2B) \sin + (180^\circ - 2A) \\ = \sin 2A + \sin 2B + \sin 2C \\ = 4 \sin A \sin B \sin C \quad (406)$$

Since

$$\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C + 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C \\ = 1 - (\cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} B) + \sin^2 \frac{1}{2} C \\ + \sin \frac{1}{2} C (\cos \frac{1}{2} (A+B) - \cos(A-B)) \\ = 1 - \cos \frac{1}{2} (A+B) \cos \frac{1}{2} (A-B) \\ + \sin^2 \frac{1}{2} C + \sin \frac{1}{2} C [\cos \frac{1}{2} (A-B) \\ - \sin \frac{1}{2}] \\ = 1 - \sin \frac{1}{2} C \cos \frac{1}{2} (A-B) + \sin^2 \frac{1}{2} C \\ - \sin \frac{1}{2} C + \sin \frac{1}{2} C \cos \frac{1}{2} (A-B) \\ = 1$$

Therefore

$$\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C = 1 - 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C \quad (407)$$

$$\text{Since } \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C$$

$$= 1 + (\cos^2 A - \sin^2 B) + \cos^2 C + \cos C [\cos(A+B) \\ + \cos(A-B)] \\ = 1 + \cos(A+B) \cos(A-B) + \cos^2 C + \cos C [-\cos C \\ + \cos(A-B)] \\ = 1 - \cos C \cos(A-B) + \cos^2 C - \cos^2 C + \cos C \\ \cos(A-B)$$

$$= 1$$

Therefore

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C \quad (408)$$

Since
obtain

$$\tan (A + B + C) = \tan 180^\circ = 0, \text{ then by (384) we}$$

$$\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0$$

Whence

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \quad (409)$$

The student will easily deduce the following:

$$\cos \frac{1}{2} A + \cos \frac{1}{2} B + \cos \frac{1}{2} C = 4 \cos \frac{1}{4} (A + B) \cos \frac{1}{4} (B + C) \cos \frac{1}{4} (A + C) \quad (410)$$

$$\tan \frac{1}{2} A \tan \frac{1}{2} B + \tan \frac{1}{2} B \tan \frac{1}{2} C + \tan \frac{1}{2} A \tan \frac{1}{2} C = 1 \quad (411)$$

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C \quad (412)$$

$$\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C = \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C \quad (413)$$

$$\frac{\tan A + \tan B + \tan C}{(\sin A \sin B \sin C)^2} = \frac{\tan \frac{1}{2} A \tan \frac{1}{2} B \tan \frac{1}{2} C}{2 \cos A \cos B \cos C} \quad (414)$$

$$\frac{\tan A}{\tan B} + \frac{\tan B}{\tan A} + \frac{\tan B}{\tan C} + \frac{\tan C}{\tan B} + \frac{\tan A}{\tan C} + \frac{\tan C}{\tan A} = \sec A \sec B \sec C - 2 \quad (415)$$

Particular case of $A + B + C$

$$\sin (A+B+C) = \sin 90^\circ = 1 \quad (416)$$

$$\cos (A+B+C) = \cos 90^\circ = 0 \quad (417)$$

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= 2 \sin (A+B) \cos (A-B) + \sin 2C \\ &= 2 \cos C [\cos A \cos B + \sin A \sin B] \\ &= 2 \cos C [2 \cos A \cos B] \\ &= 4 \cos A \cos B \cos C \end{aligned} \quad (418)$$

Since $\tan (A+B+C) = \tan 90^\circ = \infty$, then by (384) we obtain

$$\tan A \tan B + \tan B \tan C + \tan A \tan C - 1 = 0$$

Whence

$$\tan A \tan B + \tan B \tan C + \tan A \tan C = 1 \quad (419)$$

Since $\cot (A+B+C) = \cot 90^\circ = 0$, then by (385)

$$\cot A \cot B \cot C - \cot A - \cot B - \cot C = 0$$

Whence

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C \quad (420)$$

Since $\sin (A+B+C) = 1$, the formula, (382) becomes, after transposing the last term:

$$\begin{aligned} \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B \\ = \sin A \sin B \sin C + 1 \end{aligned}$$

Dividing both num. and den. by $\cos A \cos B \cos C$ gives

$$\begin{aligned} \tan A + \tan B + \tan C &= \tan A \tan B \tan C \\ &+ \sec A \sec B \sec C \end{aligned} \quad (421)$$

*Application of the preceding Formulæ to the
Reduction of Trigonometrical Expressions.*

Hitherto we have established the fundamental formulæ of Analytical Trigonometry, and derived from them a large number of secondary ones which may be of frequent use of the reduction of Trigonometric expressions from a complex form to a more simple one, or to any such one as is required to fulfil certain conditions. The number of these formulæ could be increased beyond limit; but the student is now sufficiently prepared for the elementary application of Trigonometrical values to the other branches of science to which they are connected. We shall close the present Chapter in giving a few examples of reduction, indicating the methods to be used in order to effect the proposed reduction in the first, and leaving the others to the sagacity of the student,

1. Reduce the expression $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$ to a single function?

The proposed expression may be written

$$\begin{aligned} & \frac{\sin 3A + (\sin 5A + \sin A)}{\cos 3A + (\cos 5A + \cos A)} \\ &= \frac{\sin 3A + 2 \sin 3A \cos 2A}{\cos 3A + 2 \cos 3A \cos 2A}, \text{ by (228) \& (233)} \end{aligned}$$

Factoring:

$$\begin{aligned} &= \frac{\sin 3A (1 + 2 \cos 2A)}{\cos 3A (1 + \cos 2A)} \\ &= \frac{\sin 3A}{\cos 3A} = \tan 3A. \text{ Ans.} \end{aligned}$$

2. Reduce $\cos^4 A - \sin^4 A$ to a single function?

$$\begin{aligned}
 \cos^4 A - \sin^4 A &= (\cos^2 A + \sin^2 A) (\cos^2 A - \sin^2 A) \\
 &= \cos^2 A - \sin^2 A \\
 &= \cos 2A. \text{ Ans.}
 \end{aligned}$$

3. Reduce $\sec A + \tan A$ to a single function.

$$\begin{aligned}
 \sec A + \tan A &= \frac{1 + \sin A}{\cos A} \\
 &= \frac{1 - \cos (90^\circ + A)}{\sin (90^\circ + A)} \\
 &= \tan (45^\circ + \tfrac{1}{2} A). \text{ Ans.}
 \end{aligned}$$

4. Reduce $\frac{\tan \frac{1}{2} A + \cot \frac{1}{2} A}{2}$ to a single function.

$$\begin{aligned}
 \frac{\tan \frac{1}{2} A + \cot \frac{1}{2} A}{2} &= \frac{1}{2} \left(\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} + \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A} \right) \\
 &= \frac{\sin^2 \frac{1}{2} A + \cos^2 \frac{1}{2} A}{2 \sin \frac{1}{2} A \cos \frac{1}{2} A} \\
 &= \frac{1}{\sin A} = \operatorname{cosec} A. \text{ Ans.}
 \end{aligned}$$

5. Reduce $1 + \tan A \tan \frac{1}{2} A$ to a single function.

$$\begin{aligned}
 1 + \tan A \tan \frac{1}{2} A &= 1 + \frac{\sin A}{\cos A} \times \frac{1 - \cos A}{\sin A} \\
 &= 1 + \frac{1 - \cos A}{\cos A}
 \end{aligned}$$



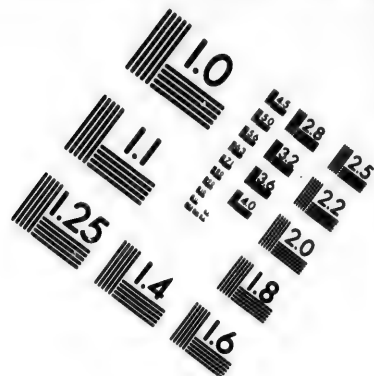
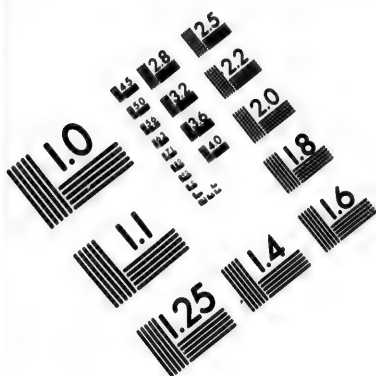
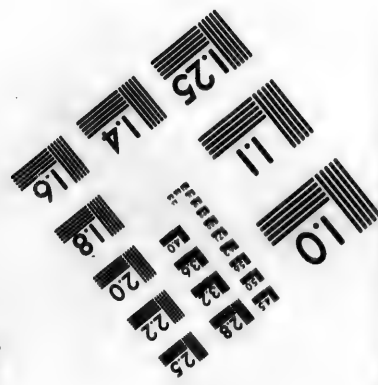
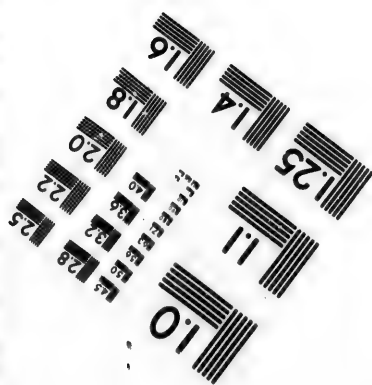
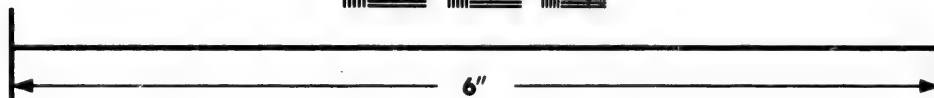
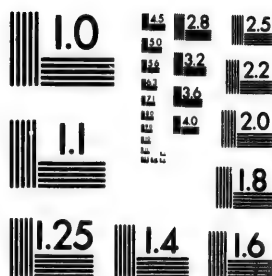
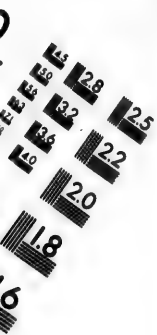


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$$= 1 + \left(\frac{1}{\cos A} - 1 \right)$$

$$= 1 + (\sec A - 1)$$

$$= \sec A. \text{ Ans.}$$

6. Reduce $\sin (30^\circ + A) + \sin (30^\circ - A)$ to a single function.
Ans. cos A

7. Reduce $\sin (45^\circ + A) \sin (45^\circ - A)$ to a single function.
Ans. $\frac{1}{2} \cos 2A$

8. Reduce $4 \sin A \sin (60^\circ + A) \sin (60^\circ - A)$ to a single function.
Ans. sin 3A

9. Reduce $\sin A \cos (A + B) - \cos A \sin (A - B)$ to an expression which can be submitted to logarithmic computation.
Ans. cos 2A sin B.

10. Reduce $\frac{\sin \frac{2}{3} A + \sin \frac{4}{3} A}{\cos \frac{2}{3} A + \cos \frac{4}{3} A}$ to a single function.
Ans. tan A.

11. Reduce $\cos^2 \frac{1}{2} A (1 + \sec \frac{1}{2} A + \tan \frac{1}{2} A) (1 - \sec \frac{1}{2} A + \tan \frac{1}{2} A)$ to a single function.
Ans. sin A.

12. Reduce $\frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A}$ to single function
Ans. $\tan^2 \frac{1}{2} A$.

13. Reduce $(\cos A + \cos B)^2 + (\sin A + \sin B)^2$ to a single function.
Ans. $4 \cos^2 \frac{1}{2} (A+B)$

14. Reduce $\frac{1 - \tan^2 (45^\circ - A)}{1 + \tan^2 (45^\circ - A)}$ to a single function.
Ans. sin 2A

15. Reduce

16. Reduce

17. Reduce

18. Reduce
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tion.

24. Reduce

25. Reduce

15. Reduce $\sin 3A \operatorname{cosec} A - \cos 3A \sec A$ to a numerical value.
Ans. 2.

16. Reduce $\frac{\cos A - \cos 3A}{\sin 3A - \sin A}$ to a single function.
Ans. $\tan 2A$.

17. Reduce $\frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A}$ to a single function.
Ans. $\tan 3A$.

18. Reduce $\cos^2 (A - B) + \cos^2 B - 2\cos (A - B) \cos A \cos B$ to a single function.
Ans. $\sin^2 A$.

19. Reduce $\sin^2 (A - B) + \sin^2 B + 2\sin (A - B) \sin B \cos A$ to a single function.
Ans. $\sin^2 A$.

20. Reduce $\frac{1}{3}(\cos^3 A \sin 3A) + \frac{1}{3}\sin^3 A \cos 3A$ to a single function.
Ans. $\frac{1}{4} \sin 4A$.

21. Reduce $\frac{\sin A + \sin mA + \sin (2m-1)A}{\cos A + \cos mA + \cos (2m-1)A}$ to a single function.
Ans. $\tan mA$.

22. Reduce $\frac{\cot \frac{1}{2} A - \tan \frac{1}{2} A}{\cot \frac{1}{2} A + \tan \frac{1}{2} A}$ to a single function.
Ans. $\cos A$.

23. Reduce $\frac{2}{\tan (45^\circ + \frac{1}{2} A) + \cot (45^\circ + \frac{1}{2} A)}$ to a single function.
Ans. $\cos A$.

24. Reduce $2\cos (45^\circ + \frac{1}{2} A) \cos (45^\circ - \frac{1}{2} A)$ to a single function.
Ans. $\cos A$.

25. Reduce $\frac{\sin 30^\circ + A - \sin (30^\circ - A)}{\sqrt{3}}$ to a single function.
Ans. $\sin A$.

26. Reduce $2 \sin^2 (45^\circ + \frac{1}{2} A) - 1$ to a single function.

Ans. $\sin A$.

27. Reduce $1 - 2 \sin^2 (45^\circ - \frac{1}{2} A)$ to a single function.

Ans. $\sin A$.

28. Reduce $\frac{\tan (45^\circ + \frac{1}{2} A) - \tan (45^\circ - \frac{1}{2} A)}{\tan (45^\circ + \frac{1}{2} A) + \tan (45^\circ - \frac{1}{2} A)}$ to single function

Ans. $\sin A$.

CHAPTER V.

INVERSE ANGULAR FUNCTIONS.

An inverse function is one in which the independent variable (see page 22) is regarded as a function of the variable which would be dependent in the corresponding direct function; so that if, taking the general functional symbol $f(x)$ we have

$$y = f(x)$$

where y is a function of x ; the corresponding inverse function is one in which x is taken as a function of y . This is expressed

$$x = f^{-1}(y)$$

and is read $x =$ a quantity whose function is y .

Hence inverse angular functions are those in which the sine, cosine, etc., are taken as the independent variable, the angles being considered as their functions. They are written according to the general notation above given. Thus, if a denote the sine of an angle, $\sin^{-1} a$ expresses the angle of which a is the sine; that is

$$\left. \begin{array}{l} \text{if } \sin A = a, \text{ we have } A = \sin^{-1} a \\ \text{" } \cos A = a, \text{ " } A = \cos^{-1} a \\ \text{" } \tan A = a, \text{ " } A = \tan^{-1} a \end{array} \right\} \quad (422)$$

And so on. They are read

$A =$ the angle whose sine is a ;

$A =$
 $A =$
And so on
Hence we
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Any relation
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Making $\cos =$
obtain

A = the angle whose cosine is a ;

A = the angle whose tangent is a

And so on.

Hence we see that an angle A , which, in $a = \sin A$ is evidently an *implicit* function of a , on account of the mutual dependence of a and A , is expressed as an *explicit* function of a by the inverse notation.

Any relation which has been established among angular functions in the preceding chapters, may be conveniently expressed by means of the *inverse* notation.

Thus by (6) we have

$$\cos A = \sqrt{1 - \sin^2 A}$$

whence by (422)

$$A = \cos^{-1} \sqrt{1 - \sin^2 A}$$

making $a = \sin A$ we have $A = \sin^{-1} a$, and it

becomes

$$\sin^{-1} a = \cos^{-1} \sqrt{1 - a^2} \quad (423)$$

In a similar way, making $\sin = a$ in (23), (33), (38), (41), we obtain easily

$$\sin^{-1} a = \operatorname{cosec}^{-1} \frac{1}{a} \quad (424)$$

$$= \tan^{-1} \frac{a}{\sqrt{1 - a^2}} \quad (425)$$

$$= \cot^{-1} \frac{\sqrt{1 - a^2}}{a} \quad (426)$$

$$\sec^{-1} \frac{1}{\sqrt{1 - a^2}} \quad (427)$$

Making $\cos = a$ in (1), (20), (34), (37), (44), (352), (367), (351), we obtain

$$\cos^{-1} a = \sin^{-1} \sqrt{1 - a^2} \quad (428)$$

$$= \sec^{-1} \frac{1}{a} \quad (429)$$

$$= \tan^{-1} \frac{\sqrt{1-a^2}}{a} \quad (430)$$

$$= \cot^{-1} \frac{a}{\sqrt{1-a^2}} \quad (431)$$

$$= \operatorname{cosec}^{-1} \frac{1}{\sqrt{1-a^2}} \quad (432)$$

$$= 2 \sin^{-1} \sqrt{\frac{1-a}{2}} \quad (433)$$

$$= 2 \cos^{-1} \sqrt{\frac{1+a}{2}} \quad (434)$$

$$= 2 \tan^{-1} \sqrt{\frac{1-a}{1+a}} \quad (435)$$

Making $\tan = a$ in (16), (27), (32), (376), (377), (185), we obtain

$$\tan^{-1} a = \cot^{-1} \frac{1}{a} \quad (436)$$

$$= \sin^{-1} \frac{a}{\sqrt{1+a^2}} \quad (437)$$

$$= \cos^{-1} \frac{1}{\sqrt{1+a^2}} \quad (438)$$

$$2 \tan^{-1} a = \sin^{-1} \frac{2a}{1+a^2} \quad (439)$$

Making $\csc$

Making $\sec$

Making cosec

By (159) we

Making $\tan$

$$(429) \quad = \tan^{-1} \frac{2a}{1-a^2} \quad (440)$$

$$(430) \quad = \cos^{-1} \left(\frac{1-a^2}{1+a^2} \right) \quad (441)$$

(431) Making $\cot = a$ in (12), (29), (30), we obtain

$$(432) \quad \cot^{-1} a = \tan^{-1} \frac{1}{a} \quad (442)$$

$$(433) \quad = \sin^{-1} \frac{1}{\sqrt{1+a^2}} \quad (443)$$

$$(434) \quad = \cos^{-1} \frac{a}{\sqrt{1+a^2}} \quad (444)$$

Making $\sec = a$ in (10), (14),

$$(435) \quad \sec^{-1} a = \cos^{-1} \frac{1}{a} \quad (445)$$

5), we obtain

$$= \tan^{-1} \sqrt{a^2-1} \quad (446)$$

Making $\operatorname{cosec} = a$ in .5, (18)

$$(436) \quad \operatorname{cosec}^{-1} a = \sin^{-1} \frac{1}{a} \quad (447)$$

$$(437) \quad = \cot^{-1} \sqrt{a^2-1} \quad (448)$$

By (159) we have

$$(438) \quad (A+B) = \tan^{-1} \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(439) Making $\tan A = m$ and $\tan B = n$, we obtain

$$\tan^{-1} m + \tan^{-1} n = \tan^{-1} \frac{m+n}{1-mn} \quad (449)$$

Similarly by (160):

$$\tan^{-1} m - \tan^{-1} n = \tan^{-1} \frac{m-n}{1+mn} \quad (450)$$

$$\text{By (161)} \quad \cot^{-1} m + \cot^{-1} n = \cot^{-1} \frac{mn-1}{m+n}$$

$$\text{By (162)} \quad \cot^{-1} m - \cot^{-1} n = \cot^{-1} \frac{mn+1}{n-m} \quad (451)$$

$$\text{By (186)} \quad 2 \tan^{-1} a = \tan^{-1} \frac{2a}{1-a^2} \quad (452)$$

$$\text{By (196)} \quad 3 \sin^{-1} a = \sin^{-1} (3a - 4a^3) \quad (453)$$

And so on for any other relation.

CHAPTER VI.

RELATIONS BETWEEN ANGULAR FUNCTIONS AND THE SIDES OF A TRIANGLE.

In this Chapter we shall investigate the relations which hold between the sides of a Triangle and the functions of its Angles. The formulæ expressing these relations are the basis of Practical Trigonometry, as will be seen in Chapter VIII, where they will be applied to the solution of Triangles.

Let the Angles of a Triangle be denoted by A, B, C , and the side respectively opposite to these angles by a, b, c . In a right-angled Triangle, A will always denote the right angle; and therefore a will

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From (454)

stand for the hypotenuse; b and c will represent the sides, including the right angle.

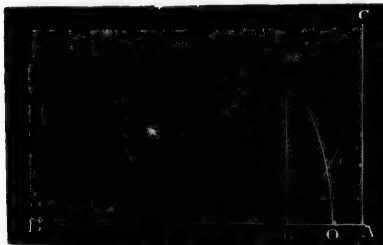
According to the method followed in this book, we shall first demonstrate a fundamental proposition from which we shall next derive analytically all the relations that may be conceived between the sides and the angles of any triangle.

Fundamental Proposition.

The radius, or unity, is to the sine of either acute angle of a right-angled triangle, in the ratio of the hypotenuse to the side opposite to that angle.

At B as centre, with a radius BO of an arbitrary length, describe the arc OP .

Let fall on AB the perpendicular PL . Then PL is the sine of the arc OP , or, (the radius being taken as unity,) of the angle B measured by the arc OP .



By the similar triangles PBL and ABC , we have

$$PP : PL :: BC : AC$$

$$\text{or} \quad 1 : \sin B :: a : b \quad (454)$$

which is the fundamental proportion above enunciated. Similarly we have

$$1 : \sin C :: a : c$$

Right-angled Triangle.

From (454) we deduce:

$$\sin B = \frac{b}{a}, \text{ or } \sin C = \frac{c}{a} \quad (455)$$

whence by (81), since B and C are complementary:

$$\cos B = \frac{c}{a}, \text{ or } \cos C = \frac{b}{a} \quad (456)$$

Dividing (455) by (456):

$$\tan B = \frac{b}{c}, \text{ or } \tan C = \frac{c}{b} \quad (457)$$

Dividing (456) by (455):

$$\cot B = \frac{c}{b}, \text{ or } \cot C = \frac{b}{c} \quad (458)$$

Taking the reciprocal of (456):

$$\sec B = \frac{a}{c}, \text{ or } \sec C = \frac{a}{b} \quad (459)$$

Taking the reciprocal of (455):

$$\operatorname{cosec} B = \frac{a}{b}, \text{ or } \operatorname{cosec} C = \frac{a}{c} \quad (460)$$

that is; in a right-angled triangle the Sine of each acute angle is equal to the opposite side divided by the hypotenuse.

The Cosine of each acute angle is equal to the adjacent side divided by the hypotenuse.

The tangent of each acute angle is equal to the opposite side divided by the adjacent one.

The Cotangent of each acute angle is equal to the adjacent side divided by the opposite one.

The Secant of each acute angle is equal to the hypotenuse divided by the adjacent side.

The Cosecant of each acute angle is equal to the hypotenuse divided by the opposite side.

Converting the above results into proportion gives, the radius being unity,

$$1 : \sin B :: a : b, \quad \text{or } 1 : \sin C :: a : c$$

$$1 : \cos B :: a : c, \quad \text{or } 1 : \cos C :: a : b$$

$$1 : \tan B :: c : b, \quad \text{or } 1 : \tan C :: b : c$$

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$$\begin{array}{ll}
 (456) \quad 1 : \cot B :: b : c, & \text{or } 1 : \cot C :: c : b \\
 1 : \sec B :: c : a, & \text{or } 1 : \sec C :: b : a \\
 1 : \operatorname{cosec} B :: b : a, & \text{or } 1 : \operatorname{cosec} C :: c : a
 \end{array}$$

From the same results we derive successively

$$\begin{array}{ll}
 (457) \quad b = a \sin B & \text{or } c = a \sin C \quad (461) \\
 = a \cos C & = a \cos B \quad (462) \\
 = c \tan B & = b \tan C \quad (463) \\
 (458) \quad = c \cot C & = b \cot B \quad (464) \\
 = \frac{a}{\sec C} & = \frac{a}{\sec B} \quad (465) \\
 (459) \quad = \frac{a}{\operatorname{cosec} B} & = \frac{a}{\operatorname{cosec} C} \quad (466)
 \end{array}$$

that is, in a right-angled triangle, either side equals:

1. the product of the hypotenuse into the sine of the opposite angle;
2. the product of the hypotenuse into the cosine of the adjacent acute angle;
3. the product of the tangent of the opposite angle into the other side;
4. the product of the cotangent of the adjacent acute angle into the other side;
5. the hypotenuse divided by the secant of the adjacent acute angle;
6. the hypotenuse divided by the cosecant of the opposite angle;

From (455), (456), (459), (460) we derive also:

$$a = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (467)$$

$$= \frac{b}{\cos C} = \frac{c}{\cos B} \quad (468)$$

$$= b \sec C = c \sec B \quad (469)$$

$$= b \operatorname{cosec} B = c \operatorname{cosec} C \quad (470)$$



that is; in a right-angled triangle, the hypotenuse equals:

1. either side divided by the sine of the angle opposite to that side;
2. either side divided by the cosine of the acute angle adjacent to that side;
3. either side multiplied by the secant of the acute angle adjacent to that side;
4. either side multiplied by the cosecant of the angle opposite to that side.

Substituting (357) in (357) gives

$$2 \sin^2 \frac{1}{2} B = 1 - \frac{c}{a} = \frac{a-c}{a}$$

whence $\sin \frac{1}{2} B = \sqrt{\left(\frac{a-c}{2a}\right)}$ (471)

Similarly $\sin \frac{1}{2} C = \sqrt{\left(\frac{a-b}{2a}\right)}$

also $a-c = 2a \sin^2 \frac{1}{2} B$ (472)

Similarly $a-b = 2a \sin^2 \frac{1}{2} C$

Substituting (456) in (371) gives

$$\tan \frac{1}{2} B = \sqrt{\left(\frac{a-c}{a+c}\right)}$$

whence $\tan \frac{1}{2} B = \frac{a-c}{b}$ (473)

$$\tan \frac{1}{2} C = \frac{a-b}{c}$$

Substituting (456) in (182) gives

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$$\cos 2B = \frac{c^2 - b^2}{a^2}$$

whence, since $2B = 90^\circ - (C - B)$,

$$\sin (C - B) = \frac{(c + b)(c - b)}{a^2}$$

and by (179)

(474)

$$\cos (C - B) = \frac{2bc}{a^2}$$

(475)

and by (475) and (476)

$$\tan (C - B) = \frac{(c + b)(c - b)}{2bc}$$

(476)

Oblique angled Triangle.

In the triangle ABC , let fall the perpendicular CP on AB , then the two right-angled triangles ACP and BCP give by (461)



$$CP = b \sin A$$

$$CP = c \sin B$$

$$a \sin B = b \sin A$$

Hence

Whence

Similarly

$$\left. \begin{aligned} a : b &= \sin A : \sin B \\ a : c &= \sin A : \sin C \\ b : c &= \sin B : \sin C \end{aligned} \right\}$$

(477)

that is; in any triangle, the sides are proportional to the sines of the opposite angles.

These three proportions being written in one, give

$$a : b : c = \sin A : \sin B : \sin C \quad (478)$$

which gives

$$\left. \begin{aligned} a &= b \cdot \frac{\sin A}{\sin B} = c \cdot \frac{\sin A}{\sin C} \\ b &= a \cdot \frac{\sin B}{\sin A} = c \cdot \frac{\sin B}{\sin C} \\ c &= a \cdot \frac{\sin C}{\sin A} = b \cdot \frac{\sin C}{\sin B} \end{aligned} \right\} \quad (479)$$

and conversely

$$\left. \begin{aligned} \sin A &= \sin B \frac{a}{b} = \sin C \frac{a}{c} \\ \sin B &= \sin A \frac{b}{a} = \sin C \frac{b}{c} \\ \sin C &= \sin A \frac{c}{a} = \sin B \frac{c}{b} \end{aligned} \right\} \quad (480)$$

(478) may, also, be written thus:

$$= \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (481)$$

These relations hold true when the perpendicular CP falls without the triangle as in ABC , because by the supplemental relation between the angle ABC and the angle PBC , we have

$$\sin ABC = \sin PBC.$$

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By Composition and division, the first equation of (477) gives

$$a + b : a - b = \sin A + \sin B : \sin A - \sin B$$

(479)

or

$$\frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B}$$

Comparing with (339):

$$\left. \begin{aligned} \frac{a + b}{a - b} &= \frac{\tan \frac{1}{2} (A + B)}{\tan \frac{1}{2} (A - B)} \end{aligned} \right\}$$

Similarly

$$\left. \begin{aligned} \frac{a + c}{a - c} &= \frac{\tan \frac{1}{2} (A + C)}{\tan \frac{1}{2} (A - C)} \end{aligned} \right\}$$

(482)

(480)

$$\left. \begin{aligned} \frac{c + b}{c - b} &= \frac{\tan \frac{1}{2} (C + B)}{\tan \frac{1}{2} (C - B)} \end{aligned} \right\}$$

Converting into proportion gives

$$\left. \begin{aligned} a + b : a - b &= \tan \frac{1}{2} (A + B) : \tan \frac{1}{2} (A - B) \\ a + c : a - c &= \tan \frac{1}{2} (A + C) : \tan \frac{1}{2} (A - C) \\ c + b : c - b &= \tan \frac{1}{2} (C + B) : \tan \frac{1}{2} (C - B) \end{aligned} \right\}$$

438

that is; in any triangle, the sum of any two sides is to their difference, in the ratio of the tangent of half the sum of the angles opposite to these sides to the tangent of half their difference.

whence the practical formulae:

(481)
 $\frac{c}{\sin C}$
 Circular CP falls with
 supplemental relation
 we have
 C.

$$\left. \begin{aligned} a - b &= (a + b) \cdot \frac{\tan \frac{1}{2} (A - B)}{\tan \frac{1}{2} (A + B)} \\ a - c &= (a + c) \cdot \frac{\tan \frac{1}{2} (A - C)}{\tan \frac{1}{2} (A + B)} \\ c - b &= (c + b) \cdot \frac{\tan \frac{1}{2} (C - B)}{\tan \frac{1}{2} (C + B)} \end{aligned} \right\} \quad (484)$$

and conversely

$$\left. \begin{aligned} \tan \frac{1}{2} (A - B) &= \tan \frac{1}{2} (A + B) \cdot \frac{a - b}{a + b} \\ \tan \frac{1}{2} (A - C) &= \tan \frac{1}{2} (A + B) \cdot \frac{a - c}{a + c} \\ \tan \frac{1}{2} (C - B) &= \tan \frac{1}{2} (C + B) \cdot \frac{c - b}{c + b} \end{aligned} \right\} \quad (485)$$

It is evident that in the last two figures we have

$$\begin{aligned} AB &= AP + PB = b \cos A + a \cos B \\ \text{or } AB &= AP - PB = b \cos A - a \cos (180^\circ - B) \\ &= b \cos A + a \cos B, \text{ by (86)} \end{aligned}$$

Hence the relation is true whether the perpendicular falls within or without the triangle; we have finally

$$\left. \begin{aligned} a &= b \cos C + c \cos B \\ b &= c \cos A + a \cos C \\ c &= a \cos B + b \cos A \end{aligned} \right\} \quad (486)$$

that is; in any triangle, each side is equal to the sum of the product of each of the others into the cosine of the angle which it makes

with the first side.

From the third equation of (486) we obtain

$$a \cos B = c - b \cos A$$

whence

$$a^2 \cos^2 B = c^2 + b^2 \cos^2 A - 2bc \cos A$$

and since by (481) $a^2 \sin^2 B = b^2 \sin^2 A$, we obtain by addition

$$a^2 (\cos^2 B + \sin^2 B) = c^2 + b^2 (\cos^2 A + \sin^2 A) - 2bc \cos A$$

whence by [a] page 13:

$$\text{Similarly } \left. \begin{aligned} a^2 &= c^2 + b^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned} \right\} \quad (487)$$

whence

$$\left. \begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned} \right\} \quad (488)$$

which express the Cosine of each angle in terms of the sides.

Adding both members of the first equation of (488) to, and subtracting them from unity:

$$\begin{aligned} 1 + \cos A &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b+c)^2 - a^2}{2bc} \end{aligned}$$

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by (86)

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(486)

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e which it makes

$$= \frac{(a+b+c)(b+c-a)}{2bc} \quad [u]$$

and $1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{[a - (b-c)][a + (b-c)]}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc} \quad [v]$$

But by [a] (page 13), $\sin^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$

$$\text{Therefore } \sin^2 A = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4b^2c^2}$$

making $a+b+c = 2s$, it becomes

$$\left. \begin{aligned} \sin^2 A &= \frac{4s(s-a)(s-b)(s-c)}{b^2c^2} \\ \text{Similarly } \sin^2 B &= \frac{4s(s-a)(s-b)(s-c)}{a^2c^2} \\ \sin^2 C &= \frac{4s(s-a)(s-b)(s-c)}{a^2b^2} \end{aligned} \right\} \quad (489)$$

$$\begin{aligned} \text{whence} \quad \left. \begin{aligned} \sin A &= \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} \\ \sin B &= \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)} \\ \sin C &= \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)} \end{aligned} \right\} \quad (490) \end{aligned}$$

which express the sine of each angle in terms of the sides.

By (357), $2 \sin^2 \frac{1}{2} A = 1 - \cos A$

$$= \frac{(a+c-b)(a+b-c)}{2bc}, \text{ by [v]}$$

making $a+b+c=2s$

$$= \frac{2(s-b)2(s-c)}{2bc} = \frac{2(s-b)(s-c)}{bc}$$

$$\begin{aligned} \text{Hence} \quad \left. \begin{aligned} \sin^2 \frac{1}{2} A &= \frac{(s-b)(s-c)}{bc} \\ \text{Similarly} \quad \sin^2 \frac{1}{2} B &= \frac{(s-a)(s-c)}{ac} \\ \text{whence} \quad \sin^2 \frac{1}{2} C &= \frac{(s-a)(s-b)}{ab} \end{aligned} \right\} \quad (491) \end{aligned}$$

$$\begin{aligned} \text{whence} \quad \left. \begin{aligned} \sin \frac{1}{2} A &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin \frac{1}{2} B &= \sqrt{\frac{(s-a)(s-c)}{ac}} \\ \sin \frac{1}{2} C &= \sqrt{\frac{(s-a)(s-b)}{ab}} \end{aligned} \right\} \quad (492) \end{aligned}$$

which express the sine of half of each angle in terms of the sides.

$$2 \cos^2 \frac{1}{2}A = 1 + \cos A$$

$$= \frac{(a+b+c)(b+c-a)}{2bc}, \text{ by [u]}$$

making $a+b+c = 2s$

$$\frac{1 \cdot 2 (s-a)}{2bc} = \frac{2s(s-a)}{bc}$$

$$\text{Hence } \cos^2 \frac{1}{2}A = \frac{s(s-a)}{bc}$$

$$\text{Similarly } \cos^2 \frac{1}{2}B = \frac{s(s-b)}{ac}$$

$$\cos^2 \frac{1}{2}C = \frac{s(s-c)}{ab}$$

(493)

$$\text{Whence } \cos \frac{1}{2}A = \sqrt{\left(\frac{s(s-a)}{bc}\right)}$$

$$\cos \frac{1}{2}B = \sqrt{\left(\frac{s(s-b)}{ac}\right)}$$

$$\cos \frac{1}{2}C = \sqrt{\left(\frac{s(s-c)}{ab}\right)}$$

(494)

which express the cosine of half of each angle in terms of the sides.

Dividing (492) by (494) gives

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$$\left. \begin{aligned} \tan \frac{1}{2}A &= \sqrt{\left(\frac{(s-b)(s-c)}{s(s-a)}\right)} \\ \tan \frac{1}{2}B &= \sqrt{\left(\frac{(s-a)(s-c)}{s(s-b)}\right)} \\ \tan \frac{1}{2}C &= \sqrt{\left(\frac{(s-a)(s-b)}{s(s-c)}\right)} \end{aligned} \right\} \quad (495)$$

which express the tangent of half of each angle in terms of the sides.

Dividing (494) by (492) gives

$$\left. \begin{aligned} \cot \frac{1}{2}A &= \sqrt{\left(\frac{s(s-a)}{(s-b)(s-c)}\right)} \\ \cot \frac{1}{2}B &= \sqrt{\left(\frac{s(s-b)}{(s-a)(s-c)}\right)} \\ \cot \frac{1}{2}C &= \sqrt{\left(\frac{s(s-c)}{(s-a)(s-b)}\right)} \end{aligned} \right\} \quad (496)$$

which express the cotangent of half of each angle in terms of the sides.

The reciprocals of (494) and (492) give

$$\left. \begin{aligned} \sec \frac{1}{2}A &= \sqrt{\left(\frac{bc}{s(s-a)}\right)} \\ \sec \frac{1}{2}B &= \sqrt{\left(\frac{ac}{s(s-b)}\right)} \\ \sec \frac{1}{2}C &= \sqrt{\left(\frac{ab}{s(s-c)}\right)} \end{aligned} \right\} \quad (497)$$

terms of the sides.

which express the secant of half of each angle in terms of the sides

$$\left. \begin{aligned} \text{And } \operatorname{cosec} \frac{1}{2}A &= \sqrt{\left(\frac{bc}{(s-b)(s-c)}\right)} \\ \operatorname{cosec} \frac{1}{2}B &= \sqrt{\left(\frac{ac}{(s-a)(s-c)}\right)} \\ \operatorname{cosec} \frac{1}{2}C &= \sqrt{\left(\frac{ab}{(s-a)(s-b)}\right)} \end{aligned} \right\} \quad (498)$$

which express the cosecant of half of each angle in terms of the sides.

From (481) we have

$$\begin{aligned} \frac{a+b}{c} &= \frac{\sin A + \sin B}{\sin C} \\ &= \frac{\sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C \cos \frac{1}{2}C} \end{aligned}$$

observing the supplemental relation of $(A+B)$ and C ; and, therefore, the complementary relation of $\frac{1}{2}(A+B)$ and $\frac{1}{2}C$.

$$= \frac{\cos \frac{1}{2}C \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B) \cos \frac{1}{2}C}$$

that is

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

Similarly

$$\frac{a+c}{b} = \frac{\cos \frac{1}{2}(A-C)}{\cos \frac{1}{2}(A+C)}$$

$$\frac{b+c}{a} = \frac{\cos \frac{1}{2}(B-C)}{\cos \frac{1}{2}(B+C)}$$

(499)

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By following a similar process, we obtain

$$\left. \begin{aligned} \frac{a-b}{c} &= \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \\ \frac{a-c}{b} &= \frac{\sin \frac{1}{2}(A-C)}{\sin \frac{1}{2}(A+B)} \\ \frac{b-c}{a} &= \frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}(B+C)} \end{aligned} \right\} \quad (500)$$

Adding unity to, or subtracting it from both members of the first equation of (499) and the first of (500), we have

$$1 + \frac{a+b}{c} = 1 + \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

$$\text{or } \frac{a+b+c}{c} = \frac{\cos \frac{1}{2}(A+B) + \cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

$$\text{and } \frac{a+b}{c} - 1 = \frac{\cos \frac{1}{2}(A-B) - \cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)}$$

$$\text{or } \frac{a+b-c}{c} = \frac{\cos \frac{1}{2}(A-B) - \cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)}$$

$$\text{also } 1 + \frac{a-b}{c} = 1 + \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

$$\text{or } \frac{c+a-b}{c} = \frac{\sin \frac{1}{2}(A+B) + \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

$$\text{and } \frac{c-a-b}{c} - 1 = \frac{\sin \frac{1}{2}(A-B) - \sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A+B)}$$

$$\text{or } \frac{c-a+b}{c} = \frac{\sin \frac{1}{2}(A+B) - \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

making $s = \frac{1}{2}(a + b + c)$ in the first members of the preceding equations; and transforming the numerators of the second members by (254) or (555), and the denominators by (81) and (82), we obtain

$$\frac{s}{c} = \frac{\cos \frac{1}{2}A \cos \frac{1}{2}B}{\sin \frac{1}{2}C} \quad (501)$$

$$\frac{s-c}{c} = \frac{\sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}C} \quad (502)$$

$$\frac{s-b}{c} = \frac{\sin \frac{1}{2}A \cos \frac{1}{2}B}{\cos \frac{1}{2}C} \quad (503)$$

$$\frac{s-a}{c} = \frac{\cos \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}C} \quad (504)$$

Performing the the same operation on the other equations of (499) and (450), we obtain

$$\frac{s}{b} = \frac{\cos \frac{1}{2}A \cos \frac{1}{2}C}{\sin \frac{1}{2}B} \quad (505)$$

$$\frac{s-b}{b} = \frac{\sin \frac{1}{2}A \sin \frac{1}{2}C}{\sin \frac{1}{2}B} \quad (506)$$

$$\frac{s-a}{b} = \frac{\sin \frac{1}{2}A \cos \frac{1}{2}C}{\cos \frac{1}{2}B} \quad (507)$$

$$\frac{s-c}{b} = \frac{\cos \frac{1}{2}A \sin \frac{1}{2}C}{\cos \frac{1}{2}B} \quad (508)$$

$$\frac{s}{a} = \frac{\cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}A} \quad (509)$$

$$\frac{s-a}{a} = \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A} \quad (510)$$

Before proceeding to the next section, it is to be observed that the ratios of the sines are equal to the ratios of the cosines, it follows that the tangents have any value

$$\frac{s-b}{a} = \frac{\sin \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A} \quad (511)$$

$$\frac{s-c}{a} = \frac{\cos \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \quad (512)$$

CHAPTER VII.

TRIGONOMETRIC TABLES.

Before proceeding to apply the formulæ deduced in the last chapter to the numerical computation of triangles, we shall explain how to find the *numerical value* of Angular Functions. In fact, the various Trigonometric Quantities, the sine, the cosine, &c., are but abstract numbers representing the comparative length of certain lines. We have already seen, in Chapter III, how to obtain the numerical value of Angular Functions in particular cases (*see pages 44, 45, and 46.*) We shall show, in this chapter, how the numbers representing the angles of all degrees of magnitude, from 1' up to 90°, can be ascertained.

When the radius is taken as the unit of measure, it is manifest that the trigonometric quantities are simply the numbers expressing the ratios of the angular functions to unity, (*see page 12.*) And as the sine and the cosine of any arc are never exceed the radius, but, on the contrary, $\sin 90^\circ$ and $\cos 0^\circ$ excepted, are all less than the radius, it follows that the Sine and Cosine of all arcs or angles included between 0° and 90° are less than unity, and, therefore, are expressed by decimal fractions.

Tangents and Cotangents which vary from 0 to infinity, may have any value greater or less than unity.

Secants and Cosecants have their value for any angle of a finite magnitude, greater than unity.

The numbers corresponding to the sine, cosine, &c., of all angles from 1' to 90°, when arranged according to the ascending order of the angles, form the *Trigonometric Tables*.

There are two principal trigonometric tables. The first contains the numerical values of sine, &c., as before stated, and is called the *Table of Natural Sines*, &c. The second contains the Logarithms of the numbers in the first table, and is called the *Table of Logarithmic Sines*, &c.

We suppose here that the student is already acquainted with the nature and the use of the common table of logarithms of numbers.

The first operation to be performed in computing a table of natural sines and cosines of all the angles differing by 1' in a quadrant, is to ascertain the value of the sine of the lowest angle, or 1'. By means of this value and some formulæ before established, the values of sines and cosines of all the other angles, are easily calculated.

The radius being unity, the semicircumference is known (see geometry) to be equal to

$$\pi = 3.141592653\dots$$

Therefore, dividing π by the number of minutes contained in the semicircumference, gives the value of one minute of arc, that is;

$$\frac{3.141592653}{180 \times 60} = 0.00029088820867\dots = K$$

which is the length of one minute of arc, and is denoted by K for brevity.

We shall now prove that no difference between the sine of one minute of arc and the arc itself, exists within the ninth place of decimals, in the above result.

It is manifest, by the inspection of the figure, page 12, that AE is greater than AP , and that AP is greater than PM ; that is, the sine of an arc not exceeding 90°, the arc itself, and its tangent are in ascending order of magnitude.

The limiting value of the ratio of an arc to its sine, when the arc is indefinitely diminished, is unity. For, in the first place, we

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$$\frac{\sin A}{A} \text{ less than } \frac{A}{A}$$

In the second place,

$$\frac{\sin A}{A} \text{ greater than } \frac{\sin A}{\tan A}$$

since $\tan A$ is greater than A , as seen before.

Therefore the value of $\frac{\sin A}{A}$ necessarily lies between

$$\frac{A}{A} \text{ and } \frac{\sin A}{\tan A} \text{ or } \cos A. \text{ see (8).}$$

But the limiting value of the first ratio, is evidently 1; and that of the second, also, is 1, since $\cos 0 = 1$. Therefore, the limiting

value of $\frac{\sin A}{A}$ must, also, be unity, that is; when A is made less than any assignable quantity, or equal to 0, we have

$$\frac{\sin A}{A} = 1$$

whence

$$\sin A = A$$

or the sine of an arc having no assignable value, is equal to the arc itself.

By assigning any finite value to A , small though it be, the ratio becomes less than 1, since $\sin A$ is less than A . And the greater is the value assigned to A , the greater will be the difference between $\sin A$ and A ; and the lesser the value assigned to A , the lesser the difference between $\sin A$ and A .

It may now be shown that the sine of an arc is greater than the arc diminished by one fourth of the cube of that arc.

For the tangent of an arc, being greater than the arc, we have

$$\tan \frac{1}{2}A = \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} \text{ greater than } \frac{1}{2}A$$

whence $\sin \frac{1}{2}A \text{ greater than } \frac{1}{2}A \cos \frac{1}{2}A$

But by (373) $\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$

Substituting $\frac{1}{2}A \cos \frac{1}{2}A$ for $\sin \frac{1}{2}A$ in this equation, it becomes

$$\sin A \text{ greater than } 2 \frac{1}{2}A \cos^2 \frac{1}{2}A$$

or $\sin A \text{ greater than } A \cos^2 \frac{1}{2}A$

or $\sin A \text{ greater than } A (1 - \sin^2 \frac{1}{2}A)$

But since an arc is greater than its sine, we have, by substituting $(\frac{1}{2}A)^2$ for $\sin^2 \frac{1}{2}A$ in the last inequality, *a fortiori*

$$\sin A \text{ greater than } A[1 - (\frac{1}{2}A)^2] \text{ or } A - \frac{1}{4}A^3 \text{ which was to be proved.}$$

Now making $A = 1'$, we have consequently,

$$\sin 1' \text{ greater than } 1' - \frac{1}{4}1'^3$$

And taking the first six decimals of K , the numerical value of one minute, increasing the sixth decimal by unity, we have *a fortiori* $\sin 1'$ greater than $[0.000290888208 \dots - \frac{1}{4}.000291^3]$; that is, performing the arithmetical operations:

$$\sin 1' \text{ greater than } 0.0002908879225 \dots = H.$$

Thus we have found two decimal fractions, K and H , between which the value of $\sin 1'$ must lie, since $\sin 1'$ is less than the first

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and so on.

$\cos 1' +$

$\cos 2' +$

$\cos 3' +$

$\cos 4' +$

and so on.

(*K*) and greater than the second (*H*); and these two fractions agree in their first eight figures, therefore, we may say without error, that

$$\sin 1' = 00029088 \dots$$

since the difference between the two members of this equation begins only beyond the decimals assumed.

If a greater approximation be required, a larger number of decimals could be assumed in the approximate value of π , or the value of one second could be found first, from which the value of one minute may be easily deduced.

The value of $\cos 1'$, is found by (6), that is;

$$\begin{aligned} \cos 1' &= \sqrt{1 - \sin 1'} \\ &= \sqrt{1 - 0.00029088} \end{aligned}$$

$$= 9999999577 \dots$$

The Sine and Cosine of $1'$ being known, we may obtain the Sine and Cosine of all angles in the quadrant, by means of the formulæ (252) and (254), which, after the transposition of the second term become

$$\sin (A + B) = 2 \sin A \cos B - \sin (A - B) \quad (513)$$

$$\cos (A + B) = 2 \cos A \cos B - \cos (A - B) \quad (514)$$

Constantly making $B = 1'$, and $A = 1', 2', 3', 4', \&c.$, successively in (513) and 514, we obtain

$$\sin (1' + 1') = \sin 2' = 2 \cos 1' \sin 1' - \sin 0 = .000581776 \dots$$

$$\sin (2' + 1') = \sin 3' = 2 \cos 1' \sin 2' - \sin 1' = .00087266 \dots$$

$$\sin (3' + 1') = \sin 4' = 2 \cos 1' \sin 3' - \sin 2' = .001163552 \dots$$

$$\sin (4' + 1') = \sin 5' = 2 \cos 1' \sin 4' - \sin 3' = .001454440 \dots$$

and so on.

$$\cos (1' + 1') = \cos 2' = 2 \cos 1' \cos 1' - \cos 0 = .999999830 \dots$$

$$\cos (2' + 1') = \cos 3' = 2 \cos 1' \cos 2' - \cos 1' = .999999619 \dots$$

$$\cos (3' + 1') = \cos 4' = 2 \cos 1' \cos 3' - \cos 2' = .999999323 \dots$$

$$\cos (4' + 1') = \cos 5' = 2 \cos 1' \cos 4' - \cos 3' = .999998942 \dots$$

and so on.

By the same process continued, we shall obtain the values of the sines and cosines of all angles differing by 1' from 0° to 90°.

The tangents, cotangents, secants and cosecants may next be calculated respectively, by the formulæ (11), (15), (20), (23).

It must be remarked that an error committed in calculating the sine or cosine of an inferior angle will entail errors in the same functions of succeeding angles. Therefore, the computist must have means of detecting and correcting errors.

The most simple means consists in calculating the sines and cosines of certain angles, by an independent process based on some established property, and comparing the results thus obtained, with those obtained by the general method. Thus, for examples, we have seen how we can find the values of the functions of 45°, 30°, and 60°.

It is not necessary to continue the process beyond 30°, because by the formulæ (283) and (286), we have

$$\begin{aligned}\sin (30^\circ + B) &= \cos B - \sin (30^\circ - B) \\ \cos (30^\circ + B) &= \cos (30^\circ - B) - \sin B\end{aligned}$$

where we see that the table may be continued beyond 30° by the simple subtraction of the sine or cosine previously found. Thus:

$$\begin{aligned}\sin 30^\circ 1' &= \cos 1' - \sin 29^\circ 59' \\ \sin 40^\circ &= \cos 10^\circ - \sin 20^\circ\end{aligned}$$

and so on.

$$\begin{aligned}\cos 30^\circ 1' &= \cos 29^\circ 59' - \sin 1' \\ \cos 40^\circ &= \cos 20^\circ - \sin 10^\circ\end{aligned}$$

and so on

Moreover, it is not necessary to continue this last process beyond 45°, since the sines and cosines of the angles above 45° will be respectively the cosines and sines of their complements below 45°.

Let it be stated that in consequence of the tedious operations which the above process requires, other methods are resorted to in the actual computation of tables, which requires Calculus. In a part later on, we shall deduce by means of Calculus infinite series very rapidly converging, from which the tables are calculated with great

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The tables of Logarithmic Sines &c., contains the logarithms of the numbers in the tables of natural sines &c. But as most of these numbers are less than unity, their logarithms would consequently have negative indices. This inconvenience is avoided by multiplying the Trigonometric Functions calculated in the above manner, by a very large number which renders them all greater than unity, so that their logarithms be positive numbers. It is evident that the same operation being performed upon all, their relative value is not altered,

The number thus employed is by common consent settled at the value of $10^{10} = 10\,000\,000\,000$; and the results give the values of the trigonometric functions as if calculated to a radius being $10\,000\,000\,000$, which are simply the logarithms of each increased by 10,

Thus the natural sine of 3° is .052336, whose logarithm is

$$-2.718800$$

to which add 10

$$8.718800$$

which is the logarithmic sine of 30° in the tables.

Of course, in calculations, we must evidently allow for that increase, and we have the following rules for practice:

1. *For each logarithmic function introduced with the positive sign into an expression, subtract 10.*
2. *For each logarithmic function introduced into an expression with the negative sign, add 10.*
3. *When the same number of logarithmic functions is introduced into an expression with both the positive and the negative signs, 10 is completely neglected.*

The reason of the above rules is evident, since, in the first case there is an excess of 10; and in the second case, there is a deficit of 10; and in the third case, an equal excess is balanced by an equal deficit. —

We have stated that the logarithmic tables may be calculated by taking the logarithms of the numbers in the natural tables, and in-

creasing the index by 10. But it will be seen hereafter that they can be calculated *independently*, that is, without the use of the tables of the Natural Functions, but by infinite series.

As to the use of the Trigonometric Tables, we refer the student to books containing them, where all explanations relating to the manner of using them, are fully given.

CHAPTER VIII.

SOLUTION OF TRIANGLES.

In every Triangle there are six elements, namely, the three sides and the three angles.

The *solution of a triangle* consists in calculating, by means of the formulæ given in Chapter VI, the unknown elements, from the relation which they bear to those which are known. When three elements are known, one of them, at least, being a side, the others can always be determined.

Right-angled Triangle.

In Right-angled Triangles, the right angle being always known, any two other parts one of which is a side being given, we can determine the rest.

All cases that may occur in practice, are reduced to the four following:

1. When the hypotenuse and one of the acute angles are given.
2. When one side and one of the acute angles are given.
3. When the hypotenuse and one side are given.
4. When the two sides are given.

By (46)
whence

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Example.

Case 1.

Given the hypotenuse a and the angle B .



By (461), $b = a \sin B$
whence $\log b = \log a + \log \sin B - 10$ (515)

By (462), $c = a \cos B$
whence $\log c = \log a + \log \cos B - 10$. (516)

By Geometry, $C = 90^\circ - B$ (517)

Example. Given $a = 275$, and $B = 57^\circ 23'$, to find b , c , C .

By 515, (516), and (517), we have

$$\begin{array}{rcl} + \log 275 & \dots\dots\dots & 2.439333 \\ + \log \sin 57^\circ 23' & \dots\dots & 9.925465 \\ - 10 & \dots\dots\dots & 10. \\ \hline = \log b & \dots\dots\dots & 2.364798, \text{ giving } b = 231.63 \end{array}$$

$$\begin{array}{rcl} + \log 275 & \dots\dots\dots & 2.439333 \\ + \log \cos 57^\circ 23' & \dots\dots & 9.731602 \\ - 10 & \dots\dots\dots & 10. \\ \hline = \log c & \dots\dots\dots & 2.170935, \text{ giving } c = 148.23 \end{array}$$

$$C = 90^\circ - 57^\circ 23' = 32^\circ 37'$$

Case 2

Given one side b , and one acute angle B .

By (464) $c = b \cot B$
 whence $\log c = \log b + \log \cot B - 10$ (518)

By (467) $a = \frac{b}{\sin B}$
 whence $\log a = \log b - \log \sin B + 10$ (519)

By Geometry, $C = 90^\circ - B$ (520)

Example. Given $b = 125$, and $B = 38^\circ 41'$, to find c , a , C .
 By (518), (519), and (520), we have

$$\begin{array}{rcl} + \log 125 & \dots\dots\dots & 2.096910 \\ + \log \cot 38^\circ 41' & \dots\dots & 10.096544 \\ - 10 & \dots\dots\dots & 10. \\ \hline = \log c & \dots\dots\dots & 2.193454, \text{ giving } c = 156.15 \\ \\ + \log 125 & \dots\dots\dots & 2.096910 \\ - \log \sin 38^\circ 41' & \dots\dots & 9.795871 \\ + 10 & \dots\dots\dots & 10. \\ \hline = \log a & \dots\dots\dots & 2.301019, \text{ giving } a = 199.50 \end{array}$$

$$C = 90^\circ - 38^\circ 41' = 51^\circ 19'$$

N.B. If b is given with the adjacent angle C , then we operate by the formulæ (113) and (115)

Case 3.

Given the hypotenuse a and the side b .

By (455) and (456), $\sin B = \cos C = \frac{b}{a}$

whence $\log \sin B = \log \cos C = \log b - \log a + 10$ (521)

By Geometry, $c = \sqrt{a^2 - b^2} = \sqrt{(a+b)(a-b)}$

whence $\log c = \frac{1}{2}[\log(a+b) + \log(a-b)]$ (522)

Example. Given $a = 6539.76$, and $b = 2936.91$, to find B, C, c

By (521) and (522) we have

$$+ \log 2936.91 \dots\dots\dots 3.467890$$

$$- \log 6539.76 \dots\dots\dots 3.815562$$

$$+ 10 \dots\dots\dots 10.$$

$$= \log \sin B = \log \cos C \dots\dots\dots 9.652328, \text{ giving}$$

$$B = 26^\circ 41' 6'', \text{ and } C = 63^\circ 19' 54''$$

And since $a + b = 9476.67$, and $a - b = 3602.85$

$$+ \log 9476.67 \dots\dots\dots 3.976656$$

$$+ \log 3602.85 \dots\dots\dots 3.556646$$

$$2)7.533302$$

$$= \log c \dots\dots\dots 2.766651, \text{ giving}$$

$$c = 5843.49$$

Case 4.

Given the two sides b and c ,

By (457) and (458), $\tan B = \cot C = \frac{b}{c}$

whence $\log \tan B = \log \cot C = \log b - \log c + 10$ (523)

By Geometry, $a = \sqrt{b^2 + c^2}$ (524)

but the last formula is not suitable for logarithmic computation.
Example. Given $b = 765$, $c = 123$, to find B , C , a .

By 523 and 524, we have

$$\begin{aligned} & + \log 765 \dots\dots\dots 2.883661 \\ & - \log 123 \dots\dots\dots 2.089905 \\ & + 10 \dots\dots\dots 10. \\ & = \log \tan B = \log \cot C = 10.793756 \text{ giving } B = 31^\circ 51' 57'' \\ & \text{and } C = 9^\circ 8' 3'' \\ & a = \sqrt{[765^2 + 123^2]} \\ & = \sqrt{[585225 + 15129]} \\ & = \sqrt{[600354]} = 774.83 \end{aligned}$$

The student may see from the above results, that in any one of the four cases of right-angled Triangles, each of the three unknown elements can be determined by a proper choice of the formula from its direct relation to the given elements. Therefore, when only one part is required to be known, it can be, in every instance, determined independently of the remaining unknown parts.

* Since the sines of angles which are nearly 90° and, also, the cosines of very small angles, differ very little from each other, as may be seen by the inspection of the tables, it follows that a small angle cannot be found with great accuracy from its cosine, and should, therefore, be determined from its sine; and conversely, an angle which is nearly 90° should be computed from its cosine. In either case, the angle is always computed with great accuracy from its tangent or cotangent. Thus when the hypotenuse a and one of the sides c are nearly equal, in which case the angle B is very small and the angle C nearly 90° , the formula (471) or (473) should be used instead of (456) for computing B .

The formula (472) may be used with advantage to compute $a - c$ when B is very small, or $a - b$ when C is very small.

By the formula 475) or 476), $C - B$ may be found with great

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Oblique Angled Triangles.

All cases that may occur in practice, are reduced to the four following.

1. When two angles and one side are given.
2. When two sides and the angle opposite to one of them are given
3. When two sides and their included angle are given.
4. When the three sides are given.

In the triangle ABC , let the angles be denoted by A, B, C , and the sides respectively opposite to them by the corresponding small letters -- a, b, c .



Case 1.

Given two angles, A, B , and one side b .

By Geometry; $C = 180^\circ - (A + B)$. (525)

By (479) $a = b \frac{\sin A}{\sin B}$

whence $\log a = \log b + \log \sin A - \log \sin B$ (526)

also, $c = b \frac{\sin C}{\sin B}$

whence $\log c = \log b + \log \sin C - \log \sin B$ (527)

N. B. If the given side be included between the two given angles, then the third angle necessarily requires to be determined, first, as in (525).

Example. Given $A = 49^{\circ} 25'$, $B = 63^{\circ} 48'$, $b = 275$, to find C, a, c

By (525), (526) and (527) we have

$$C = 180^{\circ} - 49^{\circ} 25' + 63^{\circ} 48' = 66^{\circ} 47'$$

$$\begin{aligned} + \log 275 & \dots\dots\dots 2.439333 \\ + \log \sin 49^{\circ} 25' & \dots\dots\dots 9.880505 \\ & \quad 12.319838 \\ - \log \sin 63^{\circ} 48' & \dots\dots\dots 9.952918 \\ = \log a & \dots\dots\dots 2.366920, \text{ giving } a = 232.77 \\ + \log 275 & \dots\dots\dots 2.439333 \\ + \log \sin 66^{\circ} 47' & \dots\dots\dots 9.963325 \\ & \quad 12.402658 \\ - \log \sin 63^{\circ} 48' & \dots\dots\dots 9.952918 \\ & \quad 2.449740, \text{ giving } c = 281.67 \end{aligned}$$

Case 2.

Given two sides, a and b and the angle A opposite to one of them.

$$\text{By (480)} \quad \sin B = \sin A \frac{b}{a}$$

$$\text{whence } \log \sin B = \log \sin A + \log b - \log a - 10 \quad (528)$$

$$\text{By Geometry} \quad C = 180^{\circ} - (A + B) \quad (529)$$

The angle C being determined, the third side c is easily found as in Case 1

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It must be remarked that in the present case, two different triangles may be formed, each of which will satisfy the conditions of the problem; and then the solution is ambiguous. The reason of this ambiguity is that two



different and unequal triangles may be constructed, having two sides and the angle opposite to one of these sides respectively equal in both triangles, as seen in the annexed figure, where BC and $B'C$ are made equal to each other. And since the angles B and B' are supplementary, and the sines of two supplementary arcs are equal; therefore, the sine found for B , may be the sine of either the acute angle ABC , or of its supplement, the obtuse angle $AB'C$. Hence, in all cases where it is not possible to ascertain whether the required angle should be *acute* or *obtuse* the solution is ambiguous.

It is evident that there is no ambiguity:

1. when the given angle is acute, and the side opposite this angle, is greater than the other given side, to which the required angle must be opposite. For this last angle will be less than the given one, and, therefore, *a fortiori* will be *acute*;
2. when the given angle is obtuse, because both the remaining angles must be acute.

Moreover, in practice, there is generally some circumstance to determine whether the required angle is acute or obtuse.

Example. Given $a = 145$, $b = 173.3$, $A = 41^\circ 10'$ to find B , C

By (528) we have

$$+ \log \sin 41^\circ 10' \dots\dots 9.818392$$

$$+ \log 173.3 \dots\dots\dots 2.251151$$

$$\hline 12.069543$$

$$- \log 145 \dots\dots\dots 2.161368$$

$$= \log \sin B \dots\dots\dots 9.908175, \text{ giving } B = 54^\circ 2' 22''$$

$$\text{or } B = 125^\circ 57' 38''$$

275, to find
C, a, c

= 66° 47'

2.77

81.67

to one of them.

" -- 10 (528)

(529)

c is easily found

If we take the first value

$$C = 84^{\circ} 47' 38'', \text{ and the triangle is } ABC.$$

If we take the second value

$$C = 12^{\circ} 52' 23'', \text{ and the triangle is } AB'C$$

In both cases, the third side c is determined as in case 1, by (527).

Case 3.

Given two sides, a and b , and their included angle C .

$$\text{By (485)} \quad \tan \frac{1}{2}(A - B) = \tan \frac{1}{2}(A + B) \frac{a - b}{a + b}$$

$$\text{whence} \quad \begin{aligned} \log \tan \frac{1}{2}(A - B) &= \log \tan \frac{1}{2}(A + B) \\ &+ \log(a - b) - \log(a + b) \end{aligned} \quad (530)$$

The sum $(A + B)$ is found by subtracting the given angle C from 180° . And the difference $(A - B)$ being determined by the above formula, we have, by an algebraic relation:

$$\begin{aligned} \frac{1}{2}(A + B) + \frac{1}{2}(A - B) &= A \\ \frac{1}{2}(A + B) - \frac{1}{2}(A - B) &= B \end{aligned}$$

And when A and B are thus determined, the third side is found as in the first case, by (527).

Example. Given $a = 535$, $b = 420$, $C = 53^{\circ} 8'$, to find A , B , c
 we have $\frac{1}{2}(A + B) = \frac{1}{2}(180^{\circ} - 53^{\circ} 8') = 63^{\circ} 26'$
 also, $a + b = 535 + 420 = 955$
 and $a - b = 535 - 420 = 115$

angle is ABC .

angle is $AB'C$

case 1, by

$$\begin{aligned} \text{and by (530), } + \log \tan 62^\circ 26' &\dots\dots 10.300999 \\ + \log 115 &\dots\dots\dots 2.060698 \\ &\hline &\dots\dots\dots 12.361697 \\ - \log 955 &\dots\dots\dots 2.980003 \\ = \log \frac{1}{2}(A - B) &\dots\dots\dots 9.381694, \text{ giving} \\ &\dots\dots\dots 13^\circ 32' 25'' \end{aligned}$$

$$\begin{aligned} \text{whence } B &= 63^\circ 26' + 13^\circ 32' 25'' = 76^\circ 58' 25'' \\ \text{and } C &= 63^\circ 26' - 13^\circ 32' 25'' = 49^\circ 53' 35'' \end{aligned}$$

To find the side c proceed as in the example of case 1. We may however determine c by the formula (499) from which we deduce evidently

$$c = a + b \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = (a + b) \frac{\sin \frac{1}{2}C}{\cos \frac{1}{2}(A - B)}$$

angle C .

$$\frac{a-b}{a+b}$$

$(A + B)$

(530)

Case 4.

angle C from
ed by the above

Given the three sides, a, b, c , to find the angles A, B, C .
The angle can be found by the formulae (490), (492) or (494); but the most convenient method of computing them is derived from the formulae (495), the first of which, when both numerator and denominator are multiplied by $(s - a)$, becomes

$$\tan \frac{1}{2}A = \sqrt{\left(\frac{(s-a)(s-b)(s-c)}{s(s-a)^3} \right)}$$

third side is found

$$\text{or } \tan \frac{1}{2}A = \frac{1}{s-a} \sqrt{\left(\frac{(s-a)(s-b)(s-c)}{s} \right)}$$

to find A, B, c
' 26

And denoting the second factor by k , we have

$$\tan \frac{1}{2}A = \frac{k}{s-a}$$

$$\tan \frac{1}{2}B = \frac{k}{s-b}$$

$$\tan \frac{1}{2}C = \frac{k}{s-c}$$

$$\text{whence } \log \tan \frac{1}{2}A = 10 + \log k - \log (s-a) \quad (531)$$

$$\log \tan \frac{1}{2}B = 10 + \log k - \log (s-b) \quad (532)$$

$$\log \tan \frac{1}{2}C = 10 + \log k - \log (s-c) \quad (533)$$

Example. Given $a = 150$, $b = 140$, $c = 130$, to find A , B , C .

$a = 150$	$+ \log 60 = 1.778151$
$b = 140$	$+ \log 70 = 1.845098$
$c = 130$	$+ \log 80 = 1.903090$
$2) 420$	5.526339
$s = 210$	$- \log 210 = 2.322219$
Whence $s-a = 60$	$2) 3.204120$
$s-b = 70$	$= \log k = 1.602060$
$s-c = 80$	

whence by (531), (532), and (533)

$$\begin{aligned} 10 + \log k & \dots\dots\dots 11.602060 \\ - \log 60 & \dots\dots\dots 1.778151 \\ \hline = \log \tan \frac{1}{2}A & \dots\dots\dots 9.823909, \text{ giving } \frac{1}{2}A = 33^\circ 41' 24'' \end{aligned}$$

$$\begin{aligned} 10 + \log k & \dots\dots\dots 11.602060 \\ - \log 70 & \dots\dots\dots 1.845098 \\ \hline = \log \tan \frac{1}{2}B & \dots\dots\dots 9.756962, \text{ giving } \frac{1}{2}B = 29^\circ 44' 41'' \\ 10 + \log k & \dots\dots\dots 11.602060 \\ - \log 80 & \dots\dots\dots 1.903090 \\ \hline = \log \tan \frac{1}{2}C & \dots\dots\dots 9.698970, \text{ giving } \\ & \frac{1}{2}C = 26^\circ 33' 57''.5'' \end{aligned}$$

Whence, by taking the double of these results

$$A = 67^{\circ} 22' 48''$$

$$b = 59^{\circ} 29' 23''$$

$$C = 53^{\circ} 7' 49''$$

Verification $A + B + C = 180^{\circ}$

Use of Subsidiary Angles.

When in the solution of a problem, there is met with an expression not suitable for logarithmic computation, such an expression may be made so by introducing into it an angle, which is then called a *subsidiary angle*. A subsidiary angle is, therefore, an angle which, although not immediately connected with a problem, is introduced in it by the computist to facilitate his computation.

Thus, if in Case 3, of oblique angled triangles, the side c only is required, it is desirable to have some method of computing it, independently of the two angles A and B . And as by (487) we have

$$c^2 = a^2 + b^2 - 2ab \cos C$$

this formula could be used advantageously for computing c , were it not excluding the use of logarithms. To resolve it into factors, we have by equation [a] page 13 and by (182).

$$\begin{aligned} c^2 &= (a^2 + b^2) (\cos^2 \frac{1}{2}C + \sin^2 \frac{1}{2}C) - 2ab (\cos^2 \frac{1}{2}C - \sin^2 \frac{1}{2}C) \\ &= (a + b)^2 \sin^2 \frac{1}{2}C + (a - b)^2 \cos^2 \frac{1}{2}C \end{aligned}$$

$$= (a + b)^2 \sin^2 \frac{1}{2}C \left[1 + \left(\frac{a - b}{a + b} \cot \frac{1}{2}C \right)^2 \right]$$

Assume $\tan x = \frac{a - b}{a + b} \cot \frac{1}{2}C,$ (534)

(531)
(532)
(533)

and $A, B, C.$

151
098
090
339
219
120
060

$$\frac{1}{2}A = 33^{\circ} 41' 24''$$

$$\frac{1}{2}B = 29^{\circ} 44' 41''$$

$$\frac{1}{2}C = 26^{\circ} 33' 57.5''$$

$$\begin{aligned}\text{then } c^2 &= (a+b)^2 \sin^2 \frac{1}{2}C (1 + \tan^2 x) \\ &= (a+b)^2 \sin^2 \frac{1}{2}C \sec^2 x\end{aligned}$$

$$\text{Hence } c = (a+b) \sin \frac{1}{2}C \sec x$$

$$\text{or } c = (a+b) \frac{\sin \frac{1}{2}C}{\cos x} \quad (535)$$

$$\text{whence } \log c = \log (a+b) + \log \sin \frac{1}{2}C - \log \cos x \quad (536)$$

Example. Given $a = 535$, $b = 420$, $C = 53^\circ 8'$, to find c

By (534) we have

$$\log \tan x = \log (a-b) + \log \cot \frac{1}{2}C - \log (a+b)$$

$$\begin{aligned}\text{that is } & + \log 115 \dots\dots\dots 6.060698 \\ & + \log \cot 26^\circ 34' \dots\dots 10.300999 \\ & \quad \quad \quad 12.361697 \\ & - \log 955 \dots\dots\dots 2.980003 \\ & = \log \tan x \dots\dots\dots 9.381694, \text{ giving} \\ & \quad \quad \quad x = 13^\circ 32' 26'\end{aligned}$$

Now by (536), we have

$$\begin{aligned}& + \log 955 \dots\dots\dots 2.980003 \\ & + \log \sin 26^\circ 34' \dots\dots 9.650539 \\ & \quad \quad \quad 12.630542 \\ & - \log \cos 13^\circ 32' 26'' \dots 9.987758 \\ & = \log c \dots\dots\dots 2.642784, \text{ giving} \\ & \quad \quad \quad c = 439.34\end{aligned}$$

Area of a Plane Triangle.

We shall close the present chapter by giving an expression for computing the area of a plane triangle, corresponding to the four

cases of oblique angled triangles.



Case 1. Having given one side c , and two angles A, B .

Let S = area

$h = CP$, the height. Then by Geometry we have

$$S = \frac{1}{2}ch \quad (1)$$

but by (461)

$$h = b \sin A$$

Also,

$$h = a \sin B$$

Multiplying

$$h^2 = ab \sin A \sin B \quad (2)$$

By (479)

$$a = c \frac{\sin A}{\sin C}$$

Also,

$$b = c \frac{\sin B}{\sin C}$$

Multiplying

$$ab = c^2 \frac{\sin A \sin B}{\sin^2 C} = c^2 \frac{\sin A \sin B}{\sin^2 (A+B)}$$

$$\text{Substituting in (2), } h^2 = \frac{c^2 \sin^2 A \sin^2 B}{\sin^2 (A+B)}$$

or

$$h = \frac{c \sin A \sin B}{\sin (A+B)}$$

$$\text{Substituting in (1) } S = c^2 \frac{\sin A \sin B}{2 \sin (A+B)}$$

(537)

whence $\log 2S = 2 \log c + \log \sin A + \log \sin B$
 $\quad \quad \quad - \log \sin A + B) \quad (538)$

Case 2. Having given two sides b, c , and their included angle A

and by (461) $S = \frac{1}{2}ch$
 $S = \frac{1}{2}ch \sin A \quad (539)$

whence $\log 2S = \log c + \log h + \log \sin A - 10 \quad (540)$

Case 3. Having given two sides b, c , and the angle B opposite to one of them.

In this case, it is necessary first to determine the angle included between b and c ; and then proceed as before.

Case 4. Having given the three sides a, b, c .

By (539) $S = \frac{1}{2}ch \sin A$

and substituting the value (490) of $\sin A$, we obtain

$$S = \sqrt{[s(s-a)(s-b)(s-c)]} \quad (541)$$

whence $\log S = \frac{1}{4}[\log s + \log (s-a)$
 $\quad \quad \quad + \log (s-b) + \log (s-c)] \quad (542)$

Ex. 1. Given $c = 250, A = 30^\circ 42', B = 43^\circ$, to find S

By (538)	$2 \log 250$	4.795880
	$\log \sin 30^\circ 42'$	9.708032
	$\log \sin 43^\circ$	9.833783
		24.337695
	$-\log \sin 73^\circ 42'$...	9.982183
	-10	10.
	$= \log 2S$	4.355512, giving

2S =

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Ex. 2.

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Ex. 3.

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$$2S = 22673.2$$

$$S = 11336.6$$

Ex. 2. $b = 35, c = 117, A = 27^\circ$; to find S

By (540)

$+ \log 35$		1.544068
$+ \log 117$		2.068186
$+ \log \sin 27^\circ$		9.657047
		13.269301
$- 10$		10.
$= \log 2S$		3.269301, giving
		$2S = 1859.09$
		$S = 929.54$

Ex. 3. Given $a = 505.3, b = 330.7, c = 402.5$, to find S .

$$s = 619.25$$

$$s - a = 113.95$$

$$s - b = 288.55$$

$$s - c = 216.75, \text{ whence by (542)}$$

$+ \log 619.25$		2.791866
$+ \log 113.95$		2.056714
$+ \log 288.55$		2.460221
$+ \log 216.75$		2.435959
		2) 9.644760
$= \log S$		4.822380, giving
		$S = 66432.45$

Applications to the Measurement of Distances.

By the rules given in the present chapter on the solution of triangles, we are enabled to calculate the distances of inaccessible objects. A line connecting two points lying in a horizontal plane is called *horizontal distance*, or simply, *distance*.

A line connecting two points in a vertical plane is called *height*. But as the same principles rule the calculation in both cases, there being no difference between them except that the angles to be measured are horizontal in the first and vertical in the second, we shall take the matter in a general respect, and reduce the ordinary cases that occur in practice to the three following, in each of which the process is conducted wholly by the measurement of angles and of one line, called the base.

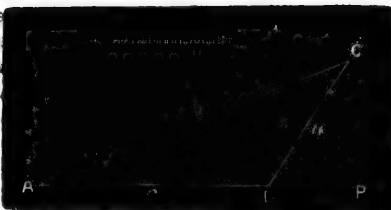
1. When one of the points connected by a required line is accessible.
2. When the two points are inaccessible, but one of them is in line with the base.
3. When the two points are inaccessible and neither in line with the base.



Case 1. Let AC be the line to be calculated, A being accessible. If the distance AB and the angle B be measured, AC is easily calculated by Case 1 of Right-angled Triangles. Thus the height of a tower is found by measuring a certain distance from its base, and observing its angular elevation at the extremity of the line measured. In the same way the width of a river is computed by measuring a certain line on its bank perpendicular at one extremity to a visual line starting from a visible object on the other bank, and observing the angle B at the other extremity of the base with the same object.

Case 2. Let CP be the line to be calculated, P being inaccessible

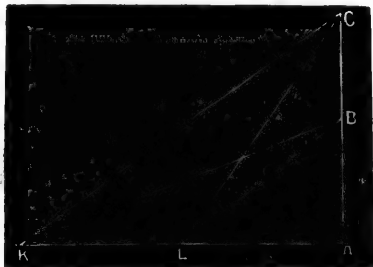
but in a line
with the
base AB . If
the length of
 AB be meas-
ured togeth-
er with the
two angles A
& CBP from



which we know the angle ABC , its supplement) then the angle ACB is known by subtracting $(A + B)$ from 180° , and AC is found from the triangle ABC by Case 1 of Oblique angled Triangles; and next CP from the triangle ABP by the Case 1 of Right-angled Triangles. Thus the height of a tower CP , the foot of which is inaccessible, but which lies in the same horizontal plane as any two stations taken in a line with it, and at any distance, is easily found by measuring the distance between the chosen stations and the angular elevations of the tower from each of them.

Case 3. Let BC be the

line to be cal-
culated, B and
 C being both
inaccessible &
not in a line
with KL . If
the visual
lines CL , and
 BL and the
angle BLC be
determined, it



is evident that BC may be calculated in the triangle BLC by the case 3 of Oblique angled Triangles. Measure the base KL , the angles CKL , BKL , CLK , and BLK ; then calculate CL

in the triangle KCL , and BL in the triangle KBL , by Case 1 of Oblique angled Triangles, and determine the angle BLC by subtracting the angle KLC from the angle KLB ; and the required elements to calculate BC are known. Thus the distance of two objects in a horizontal plane, which are inaccessible is easily found. If, however, BC be a height, as for an instance, a tower situated on a hill, it is then more simple to calculate the elevation of C above the plane KA from the triangle CKL , as in Case 2, and the elevation of the bottom B above the same plane, from the triangle BKL , and then taking the difference of the two heights thus found.

N. B. It is necessary to observe that in calculating the height of an object above a horizontal plane, we must add to the result found as before stated, that part of the height lying below the horizontal line from the eye of the observer when measuring the angular elevation.

Practical Problems.

1. A person standing at 120 feet from the base of a tower, finds its angular elevation to be $52^{\circ} 34'$; and the horizontal line from the eye when observing the last angle, intersects the tower at $5\frac{1}{2}$ feet above the ground. Calculate the height of the tower.

Ans. 168.8

2. Being on the bank of a river, I measured 463 yards by the bank; I found that a line at right angle at one extremity of the measured distance met a line on the opposite side of the river, and measuring the angle at the other extremity with the same line, I found $16^{\circ} 21'$. Required the width of the river.

Ans. 135.8

3. Two observers standing 880 feet apart, in a horizontal ground, measured at the same time, the angular elevations of a balloon whilst passing in a line with both of them, and found 35°

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and 42° .

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and 64° . Required the height of the balloon above the ground of the observation.

Ans. 935. 76 feet

4. Wanting to know the perpendicular height of a hill, I found its angular elevation, at the bottom to be $47^\circ 15'$, and 150 feet farther on a level with the bottom of the hill, I found that its angular elevation was reduced to $30^\circ 45'$. What was the required height?

Ans

5. Being on the bank of a river, and wishing to know the distance between a house and a tree on the opposite side, I took two stations on the bank at 250 feet apart; at one of them I measured the angles included between the line connecting them and the visual lines to the house and the tree, and found them to be respectively 60° and $29^\circ 36'$; at the other station, I found the angles with the house and the tree, to be 32° and $70^\circ 14'$. What was the required distance?

6. Wishing to know the height of a tower situated on a hill, I measured, from a certain point at the bottom of the hill, the angular elevations of the top of the hill and the top of the tower and found 40° and 56° ; going 200 feet further in a direct line with the tower, I found the two above angles reduced respectively to 35° and 42° . What was the height of the tower?

CHAPTER IX.

DEVELOPEMENT OF ANGULAR FUNCTIONS INTO SERIES.

As the investigation of Trigonometric Series, is more easily carried on with the aid of Differential Calculus than by a mere algebraic method, we shall give such of its elementary principles as will be

required in the following pages.

General Principle of Differentiation.

When a variable involved in any function is assumed to pass to another value, the amount of change, or the *difference* between the primitive and the new value, is called the *Increment* or *Difference* of the variable. And the amount of change which the function itself is caused to undergo by the change of the variable upon which it depends is called the *Increment* or *Difference* of the function. Thus, if in the general functional expression

$$f(x)$$

we give the increment h to the variable x , the function $f(x)$ receives a corresponding increment which is denoted by $Df(x)$, and is evidently obtained by subtracting the primitive value of the function from the value to which it is caused to pass by the increase of the variable, that is

$$Df(x) = f(x+h) - f(x) \quad (543)$$

Thus, we have

$$D \sin x = \sin(x+h) - \sin x$$

$$D \cos x = \cos(x+h) - \cos x$$

$$D \tan x = \tan(x+h) - \tan x$$

If we assume h , the increment of the variable in any function, to decrease till it become less than any assignable quantity, or infinitely diminished, it is then called the *differential of the variable*, and is symbolized by dx , if the variable is x . And the value which the increment of the function is caused to receive in this case, is called the *differential of the function*, and is symbolized by $df(x)$.

It is shown in the theory of Calculus (see my *Introduction to Differential and Integral Calculus*, page 8) that the differential of any function is always the product of the differential of the variable into a certain quantity depending upon the value of the function

and called the *Differential Coefficient*; that is, denoting that quantity by P ,

$$df(x) = Pdx, \text{ the differential}$$

whence $\frac{df(x)}{dx} = P$, the differential coefficient.

The Differential Coefficient, for any function, is obtained by dividing the increment of the function by the increment of the variable, and ascertaining the value of the quotient when the increment of the variable is made less than any assignable quantity.

Multiplying the Differential Coefficient by dx gives the Differential of the function. But the latter may also be found directly by ascertaining the value of the increment of the function, when that of the variable is made less than any assignable quantity, without making the division above mentioned.

When, for an easier notation, we make

$$y = f(x)$$

we have

$$dy = df(x), \text{ the differential}$$

and

$$\frac{dy}{dx} = \frac{df(x)}{dx}, \text{ the differential coefficient}$$

The expansion of the binomial term $f(x+h)$ in (543) assumes generally the following form:

$$f(x+h) = f(x) + Ph + Qh^2 + Rh^3 + \dots (m),$$

where the first term is the primitive function; and the other terms involve the ascending powers of h , the coefficients $P, Q, R \dots$ being independent of h , but depending upon $f(x)$. By inspecting the nature of this last expression, we see evidently, first, that the first term of the expansion will disappear by the actual subtraction of $f(x)$ indicated in (543): that is

$$f(x+h) - f(x) = Ph + Qh^2 + Rh^3 + \dots (n)$$

Therefore, any constant quantity connected by the signs + or — with $f(x)$ will disappear after differentiation, because it is evident that it should be found only in the first term of the expansion in (m) . On the contrary, any constant factor of $f(x)$ would be multiplier of the whole expression, and therefore of all of its terms, and consequently would remain factor of the differential.

Second, dividing (n) by h , the exponent of h will be diminished by unity in each term of the expansion, and will disappear in the first term. And making h less than any assignable quantity, or equal to 0, all terms but the first will vanish, and P will be the limiting value of the ratio, or the differential coefficient. Hence if $f(x+h)$ be $(x+h)^n$, or any algebraic binomial having a constant power, the second term of the expansion in (m) [or the first term in (n)] must be in all cases, by the binomial theorem, $nx^{n-1}h$; therefore $P = nx^{n-1}$; that is, if

$$y = x^n$$

$$\frac{dy}{dx} = nx^{n-1} \quad (544)$$

$$dy = nx^{n-1} dx \quad (545)$$

If $y = \log x$

$$Dy = \log(x+h) - \log(x)$$

$$= \log\left(\frac{x+h}{x}\right) = \log\left(1 + \frac{h}{x}\right)$$

But by Algebra

$$\log(1+m) = M(m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \dots)$$

where M stands for the modulus of the system of logarithms.

(n)

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n-1 h; therefore

Therefore

$$Dy = M \left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots \right)$$

$$= \frac{Mh}{x} - \frac{Mh^2}{2x^2} + \frac{Mh^3}{3x^3} - \dots$$

Hence

$$= \frac{dy}{dx} = \frac{M}{x} \quad (540)$$

and

$$dy = \frac{Mdx}{x} \quad (541)$$

If, denoting the Napierian logarithms by l , we have

$$y = lx$$

we obtain, since the modulus in this system is unity,

$$\frac{dy}{dx} = \frac{1}{x} \quad (542)$$

(544)

and

$$dy = \frac{dx}{x} \quad (543)$$

(545)

If $y = a^x$ in which a is a constant quantity and the exponent x a variable, we have, taking the Napierian logarithm of both members
 $ly = x.la$

Differentiating gives $\frac{dy}{y} = dx.la$

whence

$$dy = ydx.la = la.a^x . dx \quad (550)$$

and

$$\frac{dy}{dx} = la.a^x \quad (551)$$

If denoting by e the base of the Napierian system of logarithms we have

+ $\frac{h}{x}$

...

of logarithms.

$$y = e^x$$

we obtain, since the logarithm of the base in any system is unity

$$\frac{dy}{dx} = e^x \quad (552)$$

and

$$dy = e^x dx \quad (553)$$

Differentiation of Angular Functions.

The differentials of Angular Functions will be most easily found if we consider, 1. that when h is assumed to be dx , we have

$$\sin dx = \tan dx = dx$$

that is the sine and the tangent an arc infinitely diminished, are equal to each other and to arc itself. For dx being, when compared to finite quantity, equal to 0, we have

$$\frac{\sin dx}{\tan dx} = \cos 0 = 1$$

whence

$$\sin dx = \tan dx$$

Therefore (see page 115) $\sin dx = \tan dx = dx$

2. that dx when connected with finite quantity by the signs $+$ or $-$ must be rejected, as being then considered equal to 0; and therefore x must be substituted for $x + dx$, $x + 2dx$, $x + \frac{1}{2}dx$.

Hence, if $y = \sin x$, we have, increasing x by h ,

$$\begin{aligned} Dy &= \sin(x+h) - \sin x \\ &= 2 \cos(x + \frac{1}{2}h) \sin \frac{1}{2}h, \text{ by (229)} \end{aligned}$$

making $h = dx$:

$$dy = 2 \cos x \times \frac{1}{2}dx$$

or

$$dy = \cos x dx \quad (554)$$

and

$$\frac{dy}{dx} = \cos x \quad (555)$$

If

$$y = \cos x$$

we have

$$\begin{aligned} Dy &= \cos(x+h) - \cos x \\ &= -2 \sin(x + \frac{1}{2}h) \sin \frac{1}{2}h, \text{ by (230)} \\ dy &= -2 \sin x \times \frac{1}{2}dx \\ &= -\sin x dx \end{aligned} \quad (556)$$

whence

$$\frac{dy}{dx} = -\sin x \quad (557)$$

and

If

we have,

$$\begin{aligned} y &= \tan x \\ Dy &= \tan(x+h) - \tan x \end{aligned}$$

$$= \frac{\sin h}{\cos(x+h) \cos x}, \text{ by (236)}$$

$$= \sec(x+h) \sec x \sin h$$

$$\begin{aligned} dy &= \sec x \times \sec x \times dx \\ &= \sec^2 x dx \end{aligned} \quad (558)$$

whence

$$\frac{dy}{dx} = \sec^2 x \quad (559)$$

and

If

$$\begin{aligned} y &= \cot x \\ Dy &= \cot(x+h) - \cot x \end{aligned}$$

$$= \frac{-\sin h}{\sin(x+h) \sin x}, \text{ by (240)}$$

$$= -\operatorname{cosec}(x+h) \operatorname{cosec} x \sin h$$

whence

$$\begin{aligned} dy &= -\operatorname{cosec} x \times \operatorname{cosec} x \times dx \\ &= -\operatorname{cosec}^2 x dx \end{aligned} \quad (560)$$

and

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x \quad (561)$$

If

$$\begin{aligned} y &= \sec x \\ Dy &= \sec (x+h) - \sec x \\ &= \frac{\sin (x+\frac{1}{2}h) \sin \frac{1}{2}h}{\cos (x+h) \cos x}, \text{ by (249)} \end{aligned}$$

whence

$$dy = \frac{2 \sin x}{\cos^2 x} dx = \tan x \sec x dx \quad (562)$$

and

$$\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \tan x \sec x \quad (563)$$

If

$$\begin{aligned} y &= \operatorname{cosec} x \\ Dy &= \operatorname{cosec} (x+h) - \operatorname{cosec} x \\ &= \frac{-2 \cos (x+\frac{1}{2}h) \sin \frac{1}{2}h}{\sin (x+h) \sin x}, \text{ by (251)} \end{aligned}$$

whence

$$dy = \frac{-\cos x dx}{\sin^2 x} = -\cot x \operatorname{cosec} x dx \quad (564)$$

and

$$\frac{dy}{dx} = \frac{-\cos x}{\sin^2 x} = -\cot x \operatorname{cosec} x \quad (565)$$

= Differentiation of Inverse Angular Functions.

The *Differential Coefficients* of Inverse Angular Function are most easily obtained by the rule that *the Differential Coefficient of any inverse function is the reciprocal of that of the corresponding direct function*. This rule is proved in the *Introduction to Differential and Integral Calculus*, (page 20). Thus

$= f^{-1}(x)$, an inverse function,
then $x = F(y)$, the corresponding direct function.

and therefore, representing by P the value of the differential coef-

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cient of $x = (y)$, we have, for that of $y = f^{-1}(x)$

$$\frac{dy}{dx} = \frac{1}{P}$$

whence

$$dy = \frac{dx}{P}$$

Hence, if
we have

$$y = \sin^{-1}x \\ x = \sin y$$

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

whence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \quad (566)$$

and

$$dy = \frac{dx}{\sqrt{1 - x^2}} \quad (567)$$

If

$$y = \cos^{-1}x$$

we have

$$\frac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$$

whence

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \quad (568)$$

and

$$dy = -\frac{dx}{\sqrt{1 - x^2}} \quad (569)$$

If

$$y = \tan^{-1}x$$

we have

$$x = \tan y$$

$$\frac{dx}{dy} = 1 + \tan^2 y = 1 + x^2$$

whence

$$\frac{dy}{dx} = \frac{1}{1+x^2} \quad (570)$$

$$dy = \frac{dx}{1+x^2} \quad (571)$$

If
we have

$$y = \cot^{-1} x$$

$$x = \cot y$$

$$\frac{dx}{dy} = -(1 + \cot^2 y) = -(1 + x^2)$$

whence

$$\frac{dy}{dx} = -\frac{1}{1+x^2} \quad (572)$$

and

$$dy = -\frac{dx}{1+x^2} \quad (573)$$

If
we have

$$y = \sec^{-1} x$$

$$x = \sec y$$

$$\frac{dx}{dy} = \tan y \sec y$$

$$= \sec y \sqrt{\sec^2 y - 1}$$

$$= x \sqrt{x^2 - 1}$$

whence

$$\frac{dy}{dx} = \frac{1}{x \sqrt{x^2 - 1}} \quad (574)$$

and

$$dy = \frac{dx}{x \sqrt{x^2 - 1}} \quad (575)$$

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If
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$$y = \operatorname{cosec}^{-1} x$$

$$x = \operatorname{cosec} y$$

$$\frac{dx}{dy} = -\cot y \operatorname{cosec} y$$

$$= -\operatorname{cosec} y \sqrt{[\operatorname{cosec}^2 y - 1]}$$

$$= -x \sqrt{[x^2 - 1]}$$

whence

$$\frac{dy}{dx} = \frac{1}{x \sqrt{[x^2 - 1]}} \quad (576)$$

and

$$dy = \frac{dx}{x \sqrt{[x^2 - 1]}} \quad (567)$$

Successive Differentiation of Angular Functions.

The Differential Coefficient of any function of a variable being in general, another function of the same variable, a differential coefficient may be derived from it according to the principles already explained. This 1st differential coefficient is then called the second differential coefficient of the primitive function. In like manner, the second differential coefficient gives rise to a third and so on. The notation of these differential coefficients is as follows;

Let

$$y = f(x)$$

then

$$\frac{dy}{dx} = \text{the first differential coefficient.}$$

$$\frac{d^2y}{dx^2} = \text{the second} \quad " \quad "$$

$$\frac{d^3y}{dx^3} = \text{the third} \quad " \quad "$$

$$\frac{d^4 y}{dx^4} = \text{the fourth differential coefficient}$$

and so on.

Therefore, if

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2 y}{dx^2} = -\sin x$$

$$\frac{d^3 y}{dx^3} = -\cos x$$

$$\frac{d^4 y}{dx^4} = \sin x$$

and so on to an infinite order.

Similarly, if

$$y = \cos x$$

$$\frac{dy}{dx} = -\sin x$$

$$\frac{d^2 y}{dx^2} = -\cos x$$

$$\frac{d^3 y}{dx^3} = \sin x$$

$$\frac{d^4 y}{dx^4} = \cos x$$

and so on to an infinite order.

Taylor's theorem and Maclaurin's theorem.

Let $y = f(x)$ denote a function of x , and h any finite quantity. Then the binomial function $f(x+h)$, when expressed in terms in-

volving the integral powers of h may be supposed to be, as seen before:

$$f(x+h) = f(x) + Ph + Qh^2 + Rh^3 + \dots (p)$$

Taking the successive differential coefficients, we have

$$\frac{df(x+h)}{dx} = P + 2Qh + 3Rh^2 + \dots$$

$$\frac{d^2f(x+h)}{dx^2} = 1.2Q + 2.3Rh + \dots$$

$$\frac{d^3f(x+h)}{dx^3} = 1.2.3R + \dots$$

etc.

etc.

The above results being independent of h , they are true for all values of h . Therefore, making $h = 0$, they become, observing that all terms containing h must vanish,

$$\frac{df(x)}{dx} = \frac{dy}{dx} = P, \text{ whence } P = \frac{dy}{dx}$$

$$\frac{d^2f(x)}{dx^2} = \frac{d^2y}{dx^2} = 1.2Q, \text{ whence } Q = \frac{d^2y}{dx^2} \cdot \frac{1}{1.2}$$

$$\frac{d^3f(x)}{dx^3} = \frac{d^3y}{dx^3} = 1.2.3R, \text{ whence } R = \frac{d^3y}{dx^3} \cdot \frac{1}{1.2.3}$$

&c.

&c

Substituting these values of $P, Q, R \dots$ in (p) gives

$$f(x+h) = y + \frac{dy}{dx} \frac{h}{1} + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \frac{h^3}{1.2.3} + \dots \quad (578)$$

which is Taylor's theorem.

Make $x = 0$ in (578), and let

X_0 denote what y becomes, under this hypothesis

$$X_1 \quad " \quad " \quad \frac{dy}{dx} \quad " \quad " \quad "$$

$$X_2 \quad " \quad " \quad \frac{d^2y}{dx^2} \quad " \quad " \quad "$$

$$X_3 \quad " \quad " \quad \frac{d^3y}{dx^3} \quad " \quad " \quad "$$

and so on; we obtain

$$f(h) = X_0 + X_1 \cdot \frac{h}{1} + X_2 \cdot \frac{h^2}{1.2} + X_3 \cdot \frac{h^3}{1.2.3} + \dots$$

and substituting x for h in the last expression

$$y = f(x) = X_0 + X_1 \cdot \frac{h}{1} + X_2 \cdot \frac{h^2}{1.2} + X_3 \cdot \frac{h^3}{1.2.3} + \dots$$

which is Maclaurin's theorem.

Expansion of Angular Functions in terms of the arc.

Making $x = 0$, in $y = \sin x$, then $y = \sin 0 = 0$; and the successive differential coefficient of $\sin x$, as found in page 150 become

$$\frac{dy}{dx} = \cos 0 = 1$$

$$\frac{d^2y}{dx^2} = -\sin 0 = 0$$

$$\frac{d^3y}{dx^3} = -\cos 0 = -1$$

$$\frac{d^4y}{dx^4} = \sin 0 = 0$$

and so on to an infinite order.

Also making $x = 0$ in $y = \cos x$, then $y = \cos 0 = 1$; and the successive differential coefficients of $\cos x$ become

and so on

Substituting

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each term

$$\begin{aligned}\frac{dy}{dx} &= -\sin 0 = 0 \\ \frac{d^2y}{dx^2} &= -\cos 0 = -1 \\ \frac{d^3y}{dx^3} &= \sin 0 = 0 \\ \frac{d^4y}{dx^4} &= \cos 0 = 1\end{aligned}$$

and so on to an infinite order.

Substituting these values for X_0, X_1, X_2 , &c., in (579) we obtain, each alternate terms vanishing in both cases, and the remaining terms recurring in the order of $+1, -1$, for the sine, and $-1, +1$, for the cosine.

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.4.3.4.5.6.7} + \dots \quad (580)$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \dots \quad (581)$$

which are known as *Euler's series*.

For brevity, we shall write $\frac{1}{(2)}$ for $\frac{1}{1.2}$; $\frac{1}{(3)}$ for $\frac{1}{1.2.3}$; and so on.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{(3)} + \frac{x^5}{5} - \dots}{1 - \frac{x^2}{(3)} + \frac{x^4}{(4)} - \dots}$$

Performing the division, we see that the result will be a series containing only the odd powers of x , commencing with x , and each term having the positive sign. We may therefore assume

$$\tan x = x + A_3 x^3 + A_5 x^5 + \dots$$

Equating the second members of the last two equations, and clearing the fraction,

$$\begin{aligned} x - \frac{x^3}{(3)} + \frac{x^5}{(5)} - \dots &= \left(1 - \frac{x^2}{(2)} + \frac{x^4}{(4)} - \dots \right) \\ &\quad \times (x + A_3 x^3 + A_5 x^5 + \dots) \\ &= x + A_3 x^3 + A_5 x^5 + \dots \\ &\quad - \frac{1}{(2)} \left[\frac{x^3}{(2)} - \frac{A_3 x^5}{(2)} + \dots \right] \\ &\quad + \frac{1}{(4)} \left[\frac{x^5}{(4)} + \dots \right] + \dots \end{aligned}$$

And equating the coefficients of like terms, we obtain, after reduction

$$A_3 = \frac{2}{(3)}$$

$$A_5 = \frac{2^4}{(5)}; \text{ and so on.}$$

$$\text{Therefore, } \tan x = x + \frac{2x^3}{1 \cdot 2 \cdot 3} + \frac{2^4 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots \quad (582)$$

Dividing (581) by (582), we see that the result will be a series commencing by $\frac{1}{x}$, containing only odd powers of x , and each term having the negative sign, excepting the first. And proceeding in a similar way as before, we obtain

$$\cot x = \frac{1}{x} - \frac{2x}{1 \cdot 2 \cdot 3} + \frac{2^4 x^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dots \quad (593)$$

Expansion of arcs in terms of the sine, cosine, and tangent.

If $y = \sin x$, then $x = \sin^{-1}y \dots$ And assume

$$\sin^{-1}y = A + By + Cy^3 + Dy^4 + Ey^5 + \dots (a)$$

Differentiating both members

$$\frac{1}{\sqrt{1-y^2}} = B + 2Cy + 3Dy^2 + 4Ey^3 + \dots (b)$$

Expanding the first member of (b) by the Binomial theorem:

$$\frac{1}{\sqrt{1-y^2}} = 1 + \frac{1}{2}y^2 + \frac{1 \cdot 3}{2 \cdot 4}y^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}y^6 + \dots (c)$$

Equating the coefficients of like powers in (b) and (c), we have

$$\begin{aligned} B &= 1, & C &= 0 \\ D &= \frac{1}{1 \cdot 2 \cdot 3}, & E &= 0 \\ F &= \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, & \text{etc.} &\dots \end{aligned}$$

and making $y = 0$ in (a), we have $A = 0$.

Substituting these values in (a), we obtain

$$x = \sin^{-1}y = y + \frac{1}{2} \cdot \frac{y^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{y^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{y^7}{7} + \dots \quad (584)$$

Since, by the complementary relation of sine and cosine,
 $\cos^{-1}y = 90^\circ - \sin^{-1}y$, we have evidently

$$\cos^{-1}y = \frac{1}{2}\pi - (584) \quad (585)$$

If $y = \tan x$, then $x = \tan^{-1}y$. And assume

$$\tan^{-1}y = A + By + Cy^2 + Dy^3 + \dots \quad (a)$$

Differentiating both members

$$\frac{1}{1+y^2} = B + 2Cy + 3Dy^2 + \dots \quad (b)$$

By the binomial theorem

$$\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + y^8 - \dots \quad (c)$$

Equating the coefficients of like powers of x in the second members of (b) and (c), we have

$$\begin{array}{ll} B = 1, & C = 0 \\ D = -\frac{1}{3}, & E = 0 \\ \&c. & \&c. \end{array}$$

Making $y = 0$ in (a), since the assumed developement is true for all values of y ; then $A = 0$. Therefore by substituting these values of A, B, C, D , &c., in (a), we obtain, each alternate terms vanishing:

$$x = \tan^{-1}y = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + \dots \quad (586)$$

which is known as *Gregory's series*

Euler's Formule and De Moivre's formulæ.

Let $y = e^x$, where e denotes the base of the Naperian system of logarithms, then, by (552) we have

$$\frac{dy}{dx} = e^x, \frac{d^2y}{dx^2} = e^x, \frac{d^3y}{dx^3} = e^x; \&c \dots$$

hence, making $x = 0$, we obtain by Maclaurin's theorem

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \dots \quad (587)$$

Substituting $x\sqrt{-1}$ for x in (587) gives

$$\begin{aligned} e^{x\sqrt{-1}} &= 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3}{1.2.3}\sqrt{-1} + \frac{x^4}{1.2.3.4} + \dots \\ &= 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots + \left(x - \frac{x^3}{1.2.3} + \dots\right)\sqrt{-1} \end{aligned}$$

or, by (581) and (580):

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1} \sin x$$

Substituting $-x$ for x in the last equation (588)

$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1} \sin x \quad (589)$$

whence $\cos x = \frac{1}{2}(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})$ (590)

$$\sin x = \frac{1}{2\sqrt{-1}} (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}) \quad (591)$$

which are Euler's formulæ.

Multiplying (590) by 2, and (591) by $2\sqrt{-1}$ we obtain

$$2 \cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} \quad (592)$$

$$2\sqrt{-1} \sin x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} \quad (593)$$

Dividing (593) by (592) gives

$$\sqrt{-1} \tan x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1} \quad (594)$$

Substituting in (588) and (589) mx for x , we obtain

$$\begin{aligned} e^{m\sqrt{-1}x} &= \cos mx + \sqrt{-1} \sin mx \\ \text{and } e^{-m\sqrt{-1}x} &= \cos mx - \sqrt{-1} \sin mx \end{aligned}$$

But the first members of the two last equations are respectively equal to $(e^{x\sqrt{-1}})^m$, and $(e^{-x\sqrt{-1}})^m$, therefore the same transformation may be made on the second members; that is

$$\begin{aligned} \cos mx + \sqrt{-1} \sin mx &= (\cos x + \sqrt{-1} \sin x)^m \\ \text{and } \cos mx - \sqrt{-1} \sin mx &= (\cos x - \sqrt{-1} \sin x)^m \end{aligned}$$

which written in one equation, become

$$\cos mx + \sqrt{-1} \sin mx = (\cos x + \sqrt{-1} \sin x)^m \quad (595)$$

which is De Moivre's formula, and is true for all integral values of m .

To obtain a complete form of De Moivre's formula, we have simply to substitute $x + 2a\pi$, for x , where a denotes any integral number, hence

$$\begin{aligned} (\cos x + \sqrt{-1} \sin x)^m &= \cos (mx + 2am\pi) \\ &\quad + \sqrt{-1} \sin (mx + 2am\pi) \end{aligned} \quad (596)$$

which is true for all values of m , integral or fractional.

By the inspection of De Moivre's formula, we see that the expression $\cos x + \sqrt{-1} \sin x$ is raised to any power by the multiplication of the angle

Expansion of powers of $\sin x$ and $\cos x$.

Let $z = e^{x\sqrt{-1}}$, then we have

$$e^{nx\sqrt{-1}} = z^n; e^{-nx\sqrt{-1}} = z^{-n}; \text{ and } e^{-x\sqrt{-1}} = z^{-1};$$

$$2 \cos x = z + z^{-1} \quad (a)$$

$$2 \sqrt{-1} \sin x = z - z^{-1} \quad (b)$$

From De Moivre's formula we deduce

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$$\begin{aligned} 2 \cos nx &= z^n + z^{-n} \quad (c) \\ 2\sqrt{-1} \sin nx &= z^n - z^{-n} \quad (d) \end{aligned}$$

Also, we have by raising (a) to the n th power

$$2^n \cos^n x = (z + z^{-1})^n$$

But

$$\begin{aligned} (z + z^{-1})^n &= z^n + n z^{n-2} + \frac{n(n-1)}{1 \cdot 2} z^{n-4} + \dots \\ &\quad + \frac{n(n-1)}{1 \cdot 2} z^{-(n-4)} + n z^{-(n-2)} + z^{-n} \end{aligned}$$

which becomes, by combining terms at equal distances from the extremes

$$(z + z^{-1})^n = z^n + z^{-n} + n(z^{n-2} + z^{-(n-2)}) + \dots$$

$$+ \frac{n(n-1)}{1 \cdot 2} (z^{n-4} + z^{-(n-4)}) + \dots \quad (e)$$

that is $2^n \cos^n x = 2 \cos nx + n[2 \cos(n-2)x] + \frac{n(n-1)}{1 \cdot 2}$

$$[2 \cos(n-4)x] + \dots$$

whence dividing by 2^n ,

$$\cos^n x = \frac{1}{2^{n-1}} [\cos nx + n \cos(n-2)x + \frac{n(n-1)}{1 \cdot 2} \cos(n-4)x + \dots] \quad (597)$$

Since in the expansion of $(z + z^{-1})^n$ there are $n+1$ terms, it follows that when n is odd, $(n+1)$ will be even, and there will be no terms independent of x in (597). If on the contrary n is even, $(n+1)$ will be odd, and there must be a middle term independent of x in the series (e), which will be of the form

$$\frac{n(n-1)(n-2) \dots (\frac{1}{2}n+2)(\frac{1}{2}n+1)}{1 \cdot 2 \cdot 3 \dots (\frac{1}{2}n-1) \frac{1}{2}n}$$

and therefore a term in (597) must be found independent of x .

Example 1. $\cos^5 x = \frac{1}{32} (\cos 5x + 5 \cos 3x + 10 \cos x$

2. $\cos^4 x = \frac{1}{24} (\cos 4x + 4 \cos 2x + 6)$

In a similar way, raising (d) to the n th power, we obtain

$$\begin{aligned} 2^n (\sqrt{-1})^n \sin^n x &= (z - z^{-1})^n \\ &= z^n - n z^{n-2} + \frac{n(n-1)}{1 \cdot 2} z^{n-4} + \dots \\ &\quad + \frac{n(n-1)}{1 \cdot 2} (-z)^{-(n-2)} + n (-z)^{-(n-4)} \\ &\quad + (-z)^{-n} \dots \quad (598) \end{aligned}$$

when n is even, (598) becomes

$$2^n (\sqrt{-1})^n \sin^n x = (z^n + z^{-n}) - n(z^{n-2} + z^{-(n-2)}) + \dots \quad (599)$$

when n is odd, (598) becomes

$$2^n (\sqrt{-1})^n \sin^n x = (z^n - z^{-n}) - n(z^{n-2} + z^{-(n-2)}) + \dots \quad (600)$$

Whence we see that in the first case, the binomial $(z + z)$ having the sign $+$, the series will involve only the cosines of multiples of x and in the second case the binomial having the sign $-$ the series will involve only the sines of multiples of x ; see (a) and (b), page 158. In both cases, dividing by $2^n (\sqrt{-1})^n$ will give the expansion of $\sin^n x$.

Example 1. To expand $\sin^3 x$, we have, by (600)

$$2^3 (\sqrt{-1})^3 \sin^3 x = 2\sqrt{-1} \sin 3x - 6\sqrt{-1} \sin x$$

whence $\sin^3 x = \frac{1}{2} (\sin 3x - 3 \sin x)$

Example 2. To expand $\sin^4 x$, we have, by (599)

$$2^4 (\sqrt{-1})^4 \sin^4 x = 2 \cos 4x - 8 \cos 2x + 12$$

whence $\sin^4 x = \frac{1}{8} (\cos 4x - 4 \cos 2x + 6)$

Development of Sine and Cosine of Multiple Angles.

By De Moivre's formula we have

$$(\cos x + \sqrt{-1} \sin x)^n = \cos nx + \sqrt{-1} \sin nx$$

and $(\cos x - \sqrt{-1} \sin x)^n = \cos nx - \sqrt{-1} \sin nx$

whence $\cos nx = (\cos x + \sqrt{-1} \sin x)^n + (\cos x - \sqrt{-1} \sin x)^n$ (a)

$\sin nx = (\cos x + \sqrt{-1} \sin x)^n - (\cos x - \sqrt{-1} \sin x)^n$ (b)

Adding (a) and (b), and dividing the result by 2, we obtain

$$\cos nx = \frac{1}{2} [(\cos x + \sqrt{-1} \sin x)^n + (\cos x - \sqrt{-1} \sin x)^n]$$

Subtracting (b) from (a) and dividing the result by $\sqrt{-1}$

$$\sin nx = \frac{1}{\sqrt{-1}} [(\cos x + \sqrt{-1} \sin x)^n - (\cos x - \sqrt{-1} \sin x)^n]$$

Expanding into series, we obtain

$$\cos nx = \cos 2x - \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} x \sin^2 x$$

$$+ \frac{n(n-1)(n-3)}{1 \cdot 2 \cdot 4} \cos^{n-4} x \sin^4 x$$

$$\begin{aligned} & \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cos^{n-6} x \sin^6 x + \dots \quad 601 \\ \sin nx &= n \cos^{n-1} x \sin x - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \times \cos^{n-3} x \sin^3 x \\ &+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \cos^{n-5} x \sin^5 x \\ &\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \times \cos^{n-7} x \sin^7 x + \dots \quad 602 \end{aligned}$$

Resolution of sin x and cos x into factors.

The values $0, \pi, 2\pi, 3\pi$, etc., make $\sin nx = 0$; therefore by the Theory of Equations *see Algebra* we have

$$\sin x = ax[x^2 - \pi^2] \times [x^2 - (2\pi)^2] \times [x^2 - (3\pi)^2] \dots$$

Dividing both members by x , and decomposing each factor in the second member, we obtain

$$\begin{aligned} \frac{\sin x}{x} &= A \pi^2 \left(1 - \frac{x^2}{\pi^2}\right) \times (2\pi)^2 \left(1 - \frac{x^2}{(2\pi)^2}\right) \times (3\pi)^2 \left(1 - \frac{x^2}{(3\pi)^2}\right) \dots \\ &= A \pi^2 (2\pi)^2 (3\pi)^2 \dots \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{(2\pi)^2}\right) \left(1 - \frac{x^2}{(3\pi)^2}\right) \dots \end{aligned}$$

But if we make $x = 0$, the first member = 1, therefore the second member must also be unity and we have

$$A \pi^2 (2\pi)^2 (3\pi)^2 \dots = 1$$

$$\text{Hence } \sin x = x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{(2\pi)^2}\right) \left(1 - \frac{x^2}{(3\pi)^2}\right) \dots \quad 603$$

Similarly, the values of $\frac{1}{2}\pi, \frac{3}{2}\pi$, &c. make $\cos x = 0$. Hence

$$\cos x = A[x^2 - (\frac{1}{2}\pi)^2] \times [x^2 - (\frac{3}{2}\pi)^2] \times [x^2 - (\frac{5}{2}\pi)^2] \dots$$

$$\begin{aligned} &= A \left(\frac{1}{2}\pi\right)^2 \left(1 - \frac{(2x)^2}{\pi^2}\right) \times \left(\frac{3}{2}\pi\right)^2 \left(1 - \frac{(2x)^2}{(3\pi)^2}\right) \times \left(\frac{5}{2}\pi\right)^2 \left(1 - \frac{(2x)^2}{(5\pi)^2}\right) \dots \\ &= A \left(\frac{1}{2}\pi\right)^2 \left(\frac{3}{2}\pi\right)^2 \left(\frac{5}{2}\pi\right)^2 \dots \left(1 - \frac{(2x)^2}{\pi^2}\right) \left(1 - \frac{(2x)^2}{(3\pi)^2}\right) \left(1 - \frac{(2x)^2}{(5\pi)^2}\right) \dots \end{aligned}$$

But if we make $x = 0$, then $\cos x = 1$; and therefore the second member must also be equal to unity, and we have

$$A \left(\frac{1}{2}\pi\right)^2 \left(\frac{3}{2}\pi\right)^2 \left(\frac{5}{2}\pi\right)^2 \dots = 1$$

Therefore

$$\cos x = \left(1 - \frac{2^2 x^2}{\pi^2}\right) \left(1 - \frac{2^2 x^2}{3^2 \pi^2}\right) \left(1 - \frac{2^2 x^2}{5^2 \pi^2}\right) \dots \quad (604)$$

Computation of π .

The numerical value of π , the ratio of the circumference of a circle to its diameter, is most easily computed by means of Gregory's series (586).

We know that $\tan 45^\circ = \tan \frac{1}{2}\pi = 1$. Therefore making $x = 1$ in (586) we obtain

$$\frac{1}{2}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \quad (605)$$

But this series converges too slowly to be of any use.

Some more rapidly converging are obtained by resolving $\frac{1}{2}\pi$ into two or more arcs whose tangents are known, and developing each of these arcs by Gregory's series and taking the algebraic sum of the results. Thus if we assume

$$\frac{1}{2}\pi = \tan^{-1}m + \tan^{-1}n \quad (606)$$

we have

$$\tan \frac{1}{2}\pi = 1 = \frac{m+n}{1-mn}$$

whence

$$m = \frac{1-n}{1+n} \quad (a)$$

or

$$n = \frac{1-m}{1+m} \quad m(b)$$

Therefore giving any value to m , the corresponding value of n is easily found; or conversely.

If we take $m = \frac{1}{2}$, we find by (b), $n = \frac{1}{3}$; whence by (606) we have

$$\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

But by Gregory's series

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{7}\left(\frac{1}{2}\right)^7 + \dots$$

$$= 0.50000000 \quad - 0.04166666 \dots$$

$$+ 0.00625000 \quad - 0.00111607 \dots$$

$$- 0.00021701 \dots - 0.00004438 \dots$$

$$+ 0.0000939 \dots - 0.00000203 \dots$$

$$+ 0.0000044 \dots - 0.0000010 \dots$$

$$+ 0.0000002 \dots - 0.0000000 \dots$$

$$= 0.50647686 \dots - 0.04282924 \dots$$

$$= 0.46364762 \dots$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \frac{1}{7}\left(\frac{1}{3}\right)^7 + \dots$$

$$= 0.33333333 \dots - 0.01234567 \dots$$

$$+ 0.00082304 \dots - 0.00006532 \dots$$

$$+ 0.00000564 \dots - 0.00000051 \dots$$

$$+ 0.0000004 \dots - 0.00000000 \dots$$

$$= 0.33416205 \dots - 0.01241157 \dots$$

$$= 0.32175048 \dots$$

therefore

$$\frac{1}{4}\pi = 0.46364762 \dots + 0.32175048 \dots$$

$$= 0.78539810 \dots$$

whence

$$\pi = 3.141592,40 \text{ accurately to six places}$$

It may be taken as a general rule that, in computing an approximate value by means of infinite series, we must extend each term of the series to two places of decimals more than the number of decimals intended in the final result, and take the terms met with be

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The following is a more rapidly converging series.

We have

$$\begin{aligned}\tan^{-1} 1 - \tan^{-1} \frac{1}{5} &= \tan^{-1} \frac{2}{3} \\ \tan^{-1} \frac{2}{3} - \tan^{-1} \frac{1}{5} &= \tan^{-1} \frac{1}{17} \\ \tan^{-1} \frac{1}{17} - \tan^{-1} \frac{1}{5} &= \tan^{-1} \frac{9}{46} \\ \tan^{-1} \frac{9}{46} - \tan^{-1} \frac{1}{5} &= \tan^{-1} \frac{1}{239}\end{aligned}$$

Adding the above equations:

$$\tan^{-1} 1 - 4 \tan^{-1} \frac{1}{5} = \tan^{-1} \left(-\frac{1}{239} \right) = -\tan^{-1} \frac{1}{239}$$

whence $\frac{1}{4}\pi = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$ (607)

which is known as Machin's series.

Now by Gregory's series, taking eight decimals to keep six in the result:

$$\begin{aligned}\tan^{-1} \frac{1}{5} &= \frac{1}{5} - \frac{1}{3} \left(\frac{1}{5} \right)^3 + \frac{1}{5} \left(\frac{1}{5} \right)^5 - \frac{1}{7} \left(\frac{1}{5} \right)^7 + \dots \\ &= 0.20000000 - 0.00266666 \dots \\ &\quad + 0.00006400 - 0.00000182 \dots \\ &\quad + 0.00000005 \dots - 0.00000000 \dots \\ &= 0.20006405 \dots - 0.00266848 \dots \\ &= 0.19739557 \dots\end{aligned}$$

whence $4 \tan^{-1} \frac{1}{239} = 0.78958218 \dots$

$$\begin{aligned} \text{Also } \tan^{-1} \frac{1}{239} &= \frac{1}{239} - \frac{1}{3} \left(\frac{1}{239} \right)^3 + \dots \\ &= 0.00418410 \dots - 0.00000007 \\ &= 0.00418403 \dots \\ \text{Therefore } \frac{1}{4}\pi &= 0.78958218 \dots - 0.00418403 \\ &= 0.78539815 \dots \end{aligned}$$

whence rejecting, the last two decimals:

$$\pi = 3.141592 \dots \text{ accurately to six places.}$$

Computation of Trigonometric Tables

Natural Sines and Cosines may be calculated by the series (580) and (581), by expressing x in arc in these series, and then summing the number of terms corresponding to the number of decimal places intended.

Thus to find $\cos 10^\circ$, we have

$$\text{arc } 10^\circ = \frac{40\pi}{180} = 0.6981317 \dots$$

Substituting this value for x in (581) we obtain

$$\begin{aligned} \cos 10^\circ &= 1.00000000 - 0.01523086 \dots \\ &+ 0.00002866 \dots - 0.00000004 \dots \\ &= 1.00003866 \dots - 0.01523090 \dots \\ &= 0.98480776 \dots \end{aligned}$$

and rejecting the last two decimals as being liable to be altered by the fifth term in the series, we have, accurately to six places

$$\cos 10^\circ = .984807$$

Another method found by Euler, the inventor of the two series above mentioned, consists of making, in these series,

$$x = m \cdot \frac{1}{2}\pi$$

and then of computing constant coefficients by means of which the sines and cosines of any angle can be calculated.

To compute the logarithmic Sines and Cosines, we make $x = m \cdot \frac{1}{2}\pi$ in (580) and (581), which then become, taking the logarithms

$$\log \sin m \cdot \frac{1}{2}\pi = \log \frac{1}{2}\pi + \log m + \log \left(1 - \frac{m^2}{2^2}\right) + \log \left(1 - \frac{m^4}{4^2} + \dots\right)$$

$$\log \cos m \cdot \frac{1}{2}\pi = \log \left(1 - \frac{m^2}{1}\right) + \log \left(1 - \frac{m^4}{3^2}\right) + \log \left(1 - \frac{m^6}{5^2}\right)$$

Developing these logarithms and taking the common system, when modulus is denoted by M , we have, arranging according to the powers of m

$$\log \sin m \cdot \frac{1}{2}\pi = \log \frac{1}{2}\pi + \log m$$

$$- m^2 M \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \right)$$

$$- m^4 \frac{M}{2} \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots \right)$$

$$- m^6 \frac{M}{3} \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \dots \right)$$

&c.

$$\log \cos m \cdot \frac{1}{2}\pi = m^2 M \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$- m^4 \frac{M}{2} \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right)$$

$$- m^6 \frac{M}{3} \left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right)$$

&c.

which, after the summation of the constant numerical series, and the substitution of the values of M and $\frac{1}{2}\pi$, give series progressing according to the ascending powers of m , each term containing a constant factor of m .

But as the series of the numerical factors are slowly converging, they are made more rapidly converging by keeping some of the first terms in the preceding formulae.

It is evident that the greater the number of terms retained, the more rapidly converging will be the series for the other terms.

For more particulars we refer the student to the preface to Callet's Tables.



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PART II.

SPHERICAL TRIGONOMETRY.

OBJECT OF SPHERICAL TRIGONOMETRY

SPHERICAL TRIGONOMETRY treats of the methods of determining the several parts of a spherical triangle from having certain parts given.

A *Spherical Triangle* is that portion of the surface of the sphere bounded by arcs of three great circles; that is, of three circles whose planes pass through the centre of the sphere.

We here suppose the student to be acquainted with spherical Geometry, at least, of so much of it as is found in Loomis' Geometry, Book IX.

In the following pages, we confine our attention to such triangles only as are treated of in Geometry; that is, the *smaller* of the two parts into which the surface of a sphere is divided by the three smaller arcs joining, by pairs, three points on that surface, and which are not in the same great circle. It is evident that those triangles have every one of their parts less than 180° . The chief reason is this limitation is that, while simplifying the theory, it excludes ~~no~~ ^{no} arc or angle which is ever necessary to consider in the practical application of Spherical Trigonometry.

CHAPTER I.

RELATIONS BETWEEN THE SIDES AND THE ANGLES OF SPHERICAL TRIANGLES.

Fundamental Proposition and Formula.

In a Spherical Triangle, the cosine of any side is equal to the continued product of the cosine of the opposite angle and the sine of the other sides, together with the product of the cosines of these sides.

This proposition is called fundamental because it serves as a key to all the rest. We shall first establish it from the inspection of the annexed figure, by means of the principles of Plane Trigonometry, and, next, we shall proceed in deducing analytically the most important relations existing between Angular functions in a Spherical Triangle.

Let ABC be a Spherical Triangle, and O the centre of the Sphere. Denote the angles by A, B, C ; and the sides opposite to them respectively by a, b, c . At the point A draw AT tangent to the arc AB , and AT' tangent to the arc



AC . Then by Geometry (Loomis, Book IX, Pr. VII) the angle A of the Spherical Triangle is measured by the plane angle TAT' . — Join OB and OC . Produce OB to meet T , and OC to meet T' ; then

since the radius of the Sphere is also the radius of the circles to which belong the arcs AB and AC , OT is the secant of AB ; and OT' the secant of AC ; that is, making the radius equal to unity:

$$\begin{aligned} OT &= \sec AC = \sec b \\ OT' &= \sec AB = \sec c \\ AT &= \tan AC = \tan b \\ AT' &= \tan AB = \tan c \end{aligned}$$

Joining TT' , we have, from plane triangles $TO T'$, and TAT' , by (487) Pl. Trig.

$$\begin{aligned} TT'^2 &= OT^2 + OT'^2 - 2OT \times OT' \times \cos TOT' \\ &= \sec^2 b + \sec^2 c - 2 \sec b \sec c \cos a \quad [a] \end{aligned}$$

$$\begin{aligned} TT'^2 &= AT^2 + AT'^2 - 2AT \times AT' \times \cos TAT' \\ &= \tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A \quad [b] \end{aligned}$$

By (19) Pl. Trig. the equation [a] becomes

$$TT'^2 = 2 + \tan^2 b + \tan^2 c - 2 \sec b \sec c \cos a \quad [c]$$

Equating [b] and [c], rejecting the common quantities, transposing, and dividing by 2.

$$\sec b \sec c \cos a = \tan b \tan c \cos A + 1$$

Or, by (20) and (11) Pl. Trig. $\frac{\cos a}{\cos b \cos c} = \frac{\sin b \sin c \cos A}{\cos b \cos c} + 1$

Multiplying by $\cos b \cos c$:

$$\begin{aligned} \cos a &= \cos A \sin b \sin c + \cos b \cos c \\ \text{Similarly } \cos b &= \cos B \sin a \sin c + \cos a \cos c \\ \cos c &= \cos C \sin a \sin b + \cos a \cos b \end{aligned} \quad (1)$$

which express the fundamental proposition as above enunciated.

From (1) we obtain evidently

$$\left. \begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ \cos B &= \frac{\cos b - \cos a \cos c}{\sin a \sin c} \\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} \end{aligned} \right\} \quad (2)$$

which express the cosine of each angle in terms of the sines and cosines of the sides.

Subtracting the first formula of (2) from, and adding it to, unity we obtain

$$\begin{aligned} 1 + \cos A &= \frac{\cos a - \cos b \cos c + \sin b \sin c}{\sin b \sin c} \\ &= \frac{\cos a - \cos (b + c)}{\sin b \sin c} \\ &= \frac{2 \sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)}{\sin b \sin c} \\ \text{making } s &= \frac{1}{2}(a + b + c); \end{aligned}$$

$$= \frac{2 \sin s \sin (s - a)}{\sin b \sin c} \quad (d)$$

By a similar process we obtain

$$1 - \cos A = \frac{2 \sin (s - b) \sin (s - c)}{\sin b \sin c} \quad [e]$$

Multiplying [d] by [e]

$$1 - \cos^2 A = \frac{4 \sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\sin^2 b \sin^2 c}$$

whence, by [a] page 13,

$$\sin A = \frac{2 \sqrt{[\sin s \sin (s-a) \sin (s-b) \sin (s-c)]}}{\sin b \sin c}$$

Similarly

$$\sin B = \frac{2 \sqrt{[\sin s \sin (s-a) \sin (s-b) \sin (s-c)]}}{\sin a \sin c}$$

$$\sin C = \frac{2 \sqrt{[\sin s \sin (s-a) \sin (s-b) \sin (s-c)]}}{\sin a \sin b}$$

(3)

which express the sine of each angle in terms of the sines of the sides.

From [d] we obtain by (388) Pl. Trig.

$$\cos \frac{1}{2} A = \sqrt{\left(\frac{\sin s \sin (s-a)}{\sin b \sin c} \right)}$$

$$\cos \frac{1}{2} B = \sqrt{\left(\frac{\sin s \sin (s-b)}{\sin a \sin c} \right)}$$

$$\cos \frac{1}{2} C = \sqrt{\left(\frac{\sin s \sin (s-c)}{\sin a \sin b} \right)}$$

(4)

which express the cosine of half of each angle in terms of the sines of the sides.

From [e] we obtain by (357) Pl. Trig.

$$\sin \frac{1}{2} A = \sqrt{\left(\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c} \right)}$$

$$\sin \frac{1}{2} B = \sqrt{\left(\frac{\sin (s-a) \sin (s-c)}{\sin a \sin c} \right)}$$

$$\sin \frac{1}{2} C = \sqrt{\left(\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b} \right)}$$

(5)

which express the Sine of half of each angle in terms of the Sines of the sides.

Dividing (5) by (4) gives

$$\left. \begin{aligned} \tan \frac{1}{2} A &= \sqrt{\left(\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)} \right)} \\ \tan \frac{1}{2} B &= \sqrt{\left(\frac{\sin (s-a) \sin (s-c)}{\sin s \sin (s-b)} \right)} \\ \tan \frac{1}{2} C &= \sqrt{\left(\frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)} \right)} \end{aligned} \right\} \quad (6)$$

which express the tangent of half of each angle in terms of the Sines of the sides.

Dividing (4) by (5) gives.

$$\left. \begin{aligned} \cot \frac{1}{2} A &= \sqrt{\left(\frac{\sin s \sin (s-a)}{\sin (s-b) \sin (s-c)} \right)} \\ \cot \frac{1}{2} B &= \sqrt{\left(\frac{\sin s \sin (s-b)}{\sin (s-a) \sin (s-c)} \right)} \\ \cot \frac{1}{2} C &= \sqrt{\left(\frac{\sin s \sin (s-c)}{\sin (s-a) \sin (s-b)} \right)} \end{aligned} \right\} \quad (7)$$

which express the Cotangent of half of each angle in terms of the sines of the sides.

Let the polar triangle of ABC be $A'B'C'$, where the angles and the sides are designated respectively by A', B', C' and a', b', c' . Then since the formulæ above established are true for any spherical triangle, we have by (2)

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'}$$

But by Geometry, Loomis Book IX, Prof. IX.)

$$\begin{aligned} A' &= 180^\circ - a, & a' &= 180^\circ - A \\ B' &= 180^\circ - b, & b' &= 180^\circ - B \\ c' &= 180^\circ - c, & c' &= 180^\circ - C \end{aligned}$$

Therefore

$$(6) \quad \cos (180^\circ - a) = \frac{\cos (180^\circ - A) - \cos (180^\circ - B) \cos (180^\circ - C)}{\sin (180^\circ - B) \sin (180^\circ - C)}$$

which, by (85) and (86) Pl. Trig., becomes, changing signs in both members,

$$\left. \begin{aligned} \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C} \\ \text{Similarly} \\ \cos b &= \frac{\cos B + \cos A \cos C}{\sin A \sin C} \\ \cos c &= \frac{\cos C + \cos A \cos B}{\sin A \sin B} \end{aligned} \right\} \quad (8)$$

which express the cosine of each side in terms of the sines and cosines of the angles.

We see by the above deduction that any formula of a spherical triangle may be, by means of the polar triangle, transformed into another, in which angles take the place of sides, or conversely. Thus the formulæ (1), (3), (4), (5), (6) give respectively the following, where $S = \frac{1}{2}(A + B + C)$:

$$\left. \begin{aligned} \cos A &= \cos a \sin B \sin C - \cos B \cos C \\ \cos B &= \cos b \sin A \sin C - \cos A \cos C \\ \cos C &= \cos c \sin A \sin B - \cos A \cos B \end{aligned} \right\} \quad (9)$$

which express the cosine of each angle in terms of the opposite side and the other angles.

$$\left. \begin{aligned} \sin a &= \frac{2\sqrt{[-\cos S \cos (S-A) \cos (S-B) \cos (S-C)]}}{\sin B \sin C} \\ \sin b &= \frac{2\sqrt{[-\cos S \cos (S-A) \cos (S-B) \cos (S-C)]}}{\sin A \sin C} \\ \sin c &= \frac{2\sqrt{[-\cos S \cos (S-A) \cos (S-B) \cos (S-C)]}}{\sin A \sin B} \end{aligned} \right\} \quad (10)$$

which express the sine of each side in terms of the sines and cosines of the angles.

$$\left. \begin{aligned} \cos \frac{1}{2} a &= \sqrt{\left(\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C} \right)} \\ \cos \frac{1}{2} b &= \sqrt{\left(\frac{\cos (S-A) \cos (S-C)}{\sin A \sin C} \right)} \\ \cos \frac{1}{2} c &= \sqrt{\left(\frac{\cos (S-A) \cos (S-B)}{\sin A \sin B} \right)} \end{aligned} \right\} \quad (11)$$

which express the cosine of half of each side in terms of the sines and cosines of the angles.

$$\left. \begin{aligned} \sin \frac{1}{2} a &= \sqrt{\left(\frac{-\cos S \cos (S-A)}{\sin B \sin C} \right)} \\ \sin \frac{1}{2} b &= \sqrt{\left(\frac{-\cos S \cos (S-B)}{\sin A \sin C} \right)} \\ \sin \frac{1}{2} c &= \sqrt{\left(\frac{-\cos S \cos (S-C)}{\sin A \sin B} \right)} \end{aligned} \right\} \quad (12)$$

which express the sine of half of each side in terms of the sines and cosines of the angles.

$$\left. \begin{aligned} \cos (S-A) \\ \cos (S-B) \\ \cos (S-C) \end{aligned} \right\} \quad (10)$$

the sines and cosines

(11)

in terms of the sines

in terms of the sines and

$$\left. \begin{aligned} \tan \frac{1}{2} a &= \sqrt{\left(\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)} \right)} \\ \tan \frac{1}{2} b &= \sqrt{\left(\frac{-\cos S \cos (S-B)}{\cos (S-A) \cos (S-C)} \right)} \\ \tan \frac{1}{2} c &= \sqrt{\left(\frac{-\cos S \cos (S-C)}{\cos (S-A) \cos (S-B)} \right)} \end{aligned} \right\} \quad (13)$$

which express the tangent of half of each side in terms of the cosines of the angles.

It must be remarked that, since the sines of the angles in a Spherical Triangle is always greater than two right angles, and less than six right angles, S which denotes $\frac{1}{2}(A+B+C)$ will always exceed 90° and never reach 270° ; that is, will always lie in the second or the third quadrant. Consequently $\cos S$ is always negative, and therefore 150° Pl Trig, $-\cos S$ is always positive in the above formulae. Also since the sum of any two sides of a spherical Triangle is always greater than the third side, that is $b+c$ greater than a , we have, by the polar triangle,

$$(180^\circ - B) + (180^\circ - C) \text{ greater than } (180^\circ - A)$$

Hence

$$B+C-A \text{ less than } 180^\circ$$

whence

$$S-A \text{ less than } 90^\circ$$

Consequently, $\cos (S-A)$, and also $\cos (S-B)$, $\cos (S-C)$ are always positive.

Therefore the formulae (10), (12) and (13), although appearing under an impossible form, are always, in reality, possible.

Dividing the first equation of (3) by $\sin a$, the second by $\sin b$ and the third by $\sin c$, we obtain three equations in which the second members are identical; therefore, equating the first members, we obtain

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad (14)$$

whence

$$\left. \begin{aligned} \sin a \sin B &= \sin b \sin A \\ \sin b \sin C &= \sin c \sin B \\ \sin c \sin A &= \sin a \sin C \end{aligned} \right\} \quad (15)$$

$$\text{or} \quad \sin a : \sin b : \sin c = \sin A : \sin B : \sin C \quad (16)$$

that is; the sines of the sides are proportional to the sines of the opposite angles:

$$\begin{aligned} \text{Therefore} \quad \sin a &= \sin b \frac{\sin A}{\sin B} = \sin c \frac{\sin A}{\sin C} \\ \sin b &= \sin a \frac{\sin B}{\sin A} = \sin c \frac{\sin B}{\sin C} \\ \sin c &= \sin a \frac{\sin C}{\sin A} = \sin b \frac{\sin C}{\sin B} \end{aligned} \quad (17)$$

$$\begin{aligned} \text{Conversely} \quad \sin A &= \sin B \frac{\sin a}{\sin b} = \sin C \frac{\sin a}{\sin c} \\ \sin B &= \sin A \frac{\sin b}{\sin a} = \sin C \frac{\sin b}{\sin c} \\ \sin C &= \sin A \frac{\sin c}{\sin a} = \sin B \frac{\sin c}{\sin b} \end{aligned} \quad (18)$$

Multiplying the third of (1) by $\cos b$, and substituting the second member of the result in the first of (1), we obtain after transposing $\cos a \cos^2 b$

$$\begin{aligned} \cos a - \cos c \cos b &= \cos A \sin b \sin c + \cos C \sin a \sin b \cos b \\ \text{But } \cos a - \cos a \cos b &= \cos a (1 - \cos^2 b) = \cos a \sin^2 b \\ \text{Substituting and dividing by } \sin b, &\text{ we obtain} \end{aligned} \quad (14)$$

$$\cos a \sin b = \cos A \sin c + \cos C \sin a \cos b \quad (15)$$

Dividing each term by $\sin a$, and substituting for

$$\frac{\sin c}{\sin a} \text{ its equal } \frac{\sin C}{\sin A} \text{ (16), it becomes after reduction} \quad (16)$$

$$\cot a \sin b = \cot A \sin C + \cos C \cos b$$

sines of the

Since the last formula is general, we may interchange the letters a, b, c , and, therefore, A, B, C , and we obtain the following complete system:

$$\cot a \sin b = \cot A \sin C + \cos C \cos b \quad (17)$$

$$\cot a \sin c = \cot A \sin B + \cos B \cos c \quad (20)$$

$$\cot b \sin a = \cot B \sin C + \cos C \cos a \quad (21)$$

$$\cot b \sin c = \cot B \sin A + \cos A \cos c \quad (22)$$

$$\cot c \sin a = \cot C \sin B + \cos B \cos a \quad (23)$$

$$\cot c \sin b = \cot C \sin A + \cos A \cos b \quad (24)$$

And by the polar triangle, we obtain also the following system:

$$\cot A \sin B = \cot a \sin c - \cos c \cos B \quad (25)$$

$$\cot A \sin C = \cot a \sin b - \cos b \cos C \quad (26)$$

$$\cot B \sin A = \cot b \sin c - \cos c \cos A \quad (27)$$

$$\cot B \sin C = \cot b \sin a - \cos a \cos C \quad (28)$$

$$\cot C \sin A = \cot c \sin b - \cos b \cos A \quad (29)$$

$$\cot C \sin B = \cot c \sin a - \cos a \cos B \quad (30)$$

Napier's Analogies.

Dividing and multiplying the first equation of 6) by the second give respectively

$$\begin{aligned} \tan \frac{1}{2}A &= \frac{\sin(s-b)}{\sin(s-a)} \\ \tan \frac{1}{2}B &= \frac{\sin(s-c)}{\sin(s-a)} \end{aligned}$$

ting the second
er transposing

or $\tan \frac{1}{2}A : \tan \frac{1}{2}B :: \sin (s - b) : \sin (s - a) [f]$

and $\tan \frac{1}{2}A \tan \frac{1}{2}B = \frac{\sin (s - c)}{\sin s}$

or $1 : \tan \frac{1}{2}A \tan \frac{1}{2}B :: \sin (s - c) : \sin s [g]$

By composition and division [f] and [g] become, when written in the form of an equation.

$$\frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{\tan \frac{1}{2}A - \tan \frac{1}{2}B} = \frac{\sin (s - b) + \sin (s - a)}{\sin (s - b) - \sin (s - a)} [h]$$

$$\frac{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B}{1 + \tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{\sin s - \sin (s - c)}{\sin s + \sin (s - c)} [i]$$

By (339) Pl. Trig., the second members of [h] and [i] become respectively

$$\begin{aligned} \frac{\sin (s - b) + \sin (s - a)}{\sin (s - b) - \sin (s - a)} &= \frac{\tan \frac{1}{2}[(s - b) + (s - a)]}{\tan \frac{1}{2}[(s - b) - (s - a)]} \\ &= \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a - b)} \end{aligned}$$

$$\begin{aligned} \frac{\sin s - \sin (s - c)}{\sin s + \sin (s - c)} &= \frac{\tan \frac{1}{2}[s - (s - c)]}{\tan \frac{1}{2}[s + (s - c)]} \\ &= \frac{\tan \frac{1}{2}c'}{\tan \frac{1}{2}(a + b)} \end{aligned}$$

And by (312) and (318) Pl. Trig., the first members of [h] and [i] become

$$\frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{\tan \frac{1}{2}A - \tan \frac{1}{2}B} = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)}$$

$$\frac{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B}{1 + \tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}$$

Substituting these values in [h] and [i] we obtain

$$\frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}$$

And
$$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a+b)}$$

or
$$\sin \frac{1}{2}(A+B) : \sin \frac{1}{2}(A-B) :: \tan \frac{1}{2}c : \tan \frac{1}{2}(a-b) \quad (31)$$

and
$$\cos \frac{1}{2}(A+B) : \cos \frac{1}{2}(A-B) :: \tan \frac{1}{2}c : \tan \frac{1}{2}(a+b) \quad (32)$$

which are known as the first and second Analogies of Napier.

By the polar triangle the two preceding formulæ are transformed into

$$\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)}$$

$$\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A+B)}$$

or
$$\sin \frac{1}{2}(a+b) : \sin \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A-B) \quad (33)$$

$$\cos \frac{1}{2}(a+b) : \cos \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A+B) \quad (34)$$

which are known as the third and fourth Analogies of Napier.

Particular Case of Right-Angled Spherical Triangle.

The formulæ which have been established in the foregoing pages, assume, when applied to the Right-Angled Spherical Triangle, simpler forms, some of the terms vanishing in consequence of containing the cosine or cotangent of the angle.

Thus if A be a right angle we have $\cos A = 0$, $\cot A = 0$, and $\sin A = 1$. In this case, the first and third of (15) become

$$\sin b = \sin a \sin B \quad (35)$$

$$\sin c = \sin a \sin C \quad (36)$$

(29) and (27) become

$$\cot C = \cot b \sin b$$

$$\cot B = \cot b \sin c$$

whence

$$\sin b = \tan c \cot C \quad (37)$$

$$\sin c = \tan b \cot B \quad (38)$$

The first of (1) becomes

$$\cos a = \cos b \cos c \quad (39)$$

The first of (9) after the transposition of the last term and the division by $\sin B \sin C$, becomes

$$\cos a = \cot B \cot C \quad (40)$$

The second and third of (9) become

$$\cos B = \cos b \sin C \quad (41)$$

$$\cos C = \cos c \sin B \quad (42)$$

(20) being divided by $\cos c$, and (19) by $\cos b$, become

$$\cos B = \cot a \tan c \quad (43)$$

$$\cos C = \cot a \tan b \quad (44)$$

If we consider the ten equations above established, we see 1st, that the first members all contain either a sine or a cosine; and the second members contain either a product of a tangent into a cotangent, or a product whose factors are sines or cosines; 2ndly, that in the first members, the two sides including the right angle, are always expressed by their sines, and the other three parts are always expressed by their cosines; and in the second members the two sides of the right angles are always expressed by their cosines or tangents, and the other three parts by their sines or

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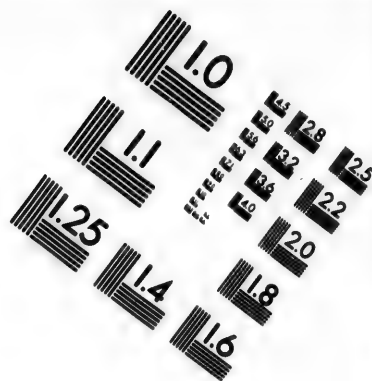
cotangents. Therefore, if we replace the cosines by sines in the first members, the sines by cosines and the cotangents by tangents in the second members, which can be most easily done by simply taking the complements of the parts a, B, C instead of the parts themselves, we shall obtain a group of equations all having a sine in the first member, and a product of either two tangents or two cosines in the second, as may be seen by the inspection the following table, where a, B', C' denote the complements of a, B, C .

$\sin b = \sin a \sin B$	$\sin b = \cos a' \cos B'$
$\sin c = \sin a \sin C$	$\sin c = \cos a' \cos C'$
$\sin b = \tan c \cot C'$	$\sin b = \tan c \tan C''$
$\sin c = \tan b \cot B$	$\sin c = \tan b \tan B'$
$\cos a = \cos b \cos c$	$\sin a = \cos b \cos c$
$\cos a = \cot B \cot C$	$\sin a = \tan B' \tan C'$
$\cos B = \cos b \sin C$	$\sin B' = \cos b \cos C'$
$\cos C = \cos c \sin B$	$\sin C' = \cos c \cos B'$
$\cos B = \cot a \tan c$	$\sin B' = \tan a' \tan c$
$\cos C = \cot a \tan b$	$\sin C' = \tan a' \tan b$

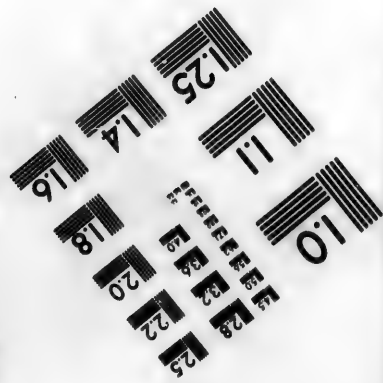
The right angle being thrown altogether out of consideration it remains in the triangle five parts, any one of which has consequently two parts on one side and two on the other side. Any one of the five parts may be assumed as the *middle* one; and then the two parts which are immediately adjacent to the right and left of it, are called *adjacent parts*; the other two are called *opposite parts*.

If now we consider the above results in connexion with the last figure, observing that a, B, C , are replaced by their complements, we see that when the sine of any part is equal to the product of the tangent of two other parts, and when the sine of any part is equal to the product of the cosines of two other parts, these last parts are always *opposite* with respect to the first. Hence the most important following rule, called Napier's rule, from the name of the inventor, and which is sufficient to solve all the cases that may occur in the practice in right-angled spherical triangles.





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Napier's Rule

The sine of any part in a right-angled Spherical Triangle is equal to the product of the tangents of the two adjacent parts, or of the cosines of the two opposite parts,

Oblique Angled Triangle and Perpendicular.

If a perpendicular be drawn from any angle in an Oblique Angled Spherical Triangle, some other useful formulae may be obtained. Thus let CP be perpendicular on AB , in the annexed figure. Denote the segment AP by v and the angle ACP by V , then the other segment $BP = AB - AP = c - v$, the angle $BCP = C - V$.

By (43) we have in the right-angled Triangle ACP

$$\cos A = \cot b \tan v$$

whence, since tangent and cotangent are mutually reciprocal,



$$\tan v = \tan b \cos A$$

$$\text{Similarly } \tan (c - v) = \tan c \cos B \quad (45)$$

that is: the tangent of either segment of the base is equal to the product of the tangent of the adjacent side into the cosine of the adjacent angle.

$$\text{By (40),} \quad \cot a = \cot B \cot V$$

$$\text{whence} \quad \cot V = \frac{\cot a}{\cot B}$$

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Taking the reciprocal,

$$\left. \begin{aligned} \tan V &= \frac{\cot B}{\cot a} \\ \text{Similarly } \tan (C-V) &= \frac{\cot A}{\cot b} \end{aligned} \right\}$$

(46)

that is *the tangent of either partial angle is equal to the cotangent of the adjacent angle divided by the adjacent side.*

By (39),

$$\begin{aligned} \cos b &= \cos v \cos CP \\ \cos a &= \cos (c-v) \cos CP \\ \text{whence } \frac{\cos b}{\cos a} &= \frac{\cos v}{\cos (c-v)} \end{aligned}$$

(47)

that is, *the cosines of the segments are proportional to the cosines of the sides adjacent to these segments.*

By (37),

$$\begin{aligned} \sin v &= \cot A \tan CP \\ \sin (c-v) &= \cot B \tan CP \end{aligned}$$

whence

$$\frac{\sin v}{\sin (c-v)} = \frac{\cot A}{\cot B} = \frac{\tan B}{\tan A}$$

(48)

that is; *the sines of the segments are directly proportional to the cotangents, or inversely proportional to the tangents of the sides adjacent to these segments*

By (41),

$$\begin{aligned} \cos A &= \sin V \cos CP \\ \cos B &= \sin (C-V) \cos CP \end{aligned}$$

whence

$$\frac{\cos A}{\cos B} = \frac{\sin V}{\sin (C-V)}$$

(49)

that is; *the sines of the partial angles made by the perpendicular are proportional to the cosines of the angles at the base.*

By (44), $\cos V = \cot b \tan CP$
 $\cos (C - V) = \cot c \tan CP$

whence
$$\frac{\cos V}{\cos (C - V)} = \frac{\cot b}{\cot c} = \frac{\tan a}{\tan b} \quad (50)$$

that is, *the cosines of the partial angles made by the perpendicular are directly proportional to the cotangents, or inversely proportional to the adjacent sides.*

By following a similar process, we deduce, comparing the two partial triangles,

$$\frac{\tan v}{\tan (c - v)} = \frac{\tan V}{\tan (C - V)}$$

that is, *the tangents of the segments of the base are proportional to the tangents of the partial angles.*

CHAPTER II.

Solution of Spherical Triangles.

If any three of the parts of a Spherical Triangle be given, the other may be determined, and it is not necessary that one of the given parts be a side, as in Plane Trigonometry.

Right-Angled Spherical Triangle.

When the rule of Napier is made use of in solving a right-angled Spherical Triangle, all the cases that may occur in practice, are found included in the following one where the right angle is not taken into consideration, as being always known:

Being given any two parts, to determine the three other parts

For the two given parts are either adjacent to each other, or separated by another part. And in every case, they must be either adjacent or opposite to, at least, one of the unknown parts which being determined may serve to determine the others.

It must be observed that when the quantity sought is determine

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The qua

ed by its sine as is always the case in the use of Napier's rule, it will be ambiguous whether the quantity is less or greater than 90° , since the sines are positive in both the first and second quadrants. The ambiguity may however generally be removed by the following principles.

I. *In a Right-Angled Spherical Triangle, an oblique angle and its opposite side are always of the same species, that is, are either both lesser or both greater than 90° .*

II. *When the two sides including the right angle are of the same species, the hypotenuse is less than 90° ; and when they are of different species, the hypotenuse is greater than 90° .*

The first of these principles follows from (37), where, since b is positive, $\tan c$ and $\cot C$ must have the same sign, and, therefore, c and C must be in the same quadrant.

The second principle follows from (39), where, if b and c are in the same quadrant, their cosines have the same sign, and therefore $\cos a$ is positive, that is, a is less than 90° : if on the contrary, b and c are in different quadrants, their cosines have different signs, and therefore $\cos a$ is negative: that is, a is greater than 90° .

Moreover, another form may be given to the rule of Napier, so that it leaves no ambiguity. For this purpose, instead of taking in the group on page 183, the sines for cosines in the first members of the equations, the tangents and the cosines for cotangents and sines in the second members, take the cosines for sines in the first members, and the cotangents and sines for tangents and cosines in the second members. Thus the five parts, instead of being, the two sides including the right angle and the complements of the other three parts, are, conversely, the complements of the two sides including the right angle, and the three other parts in their primitive condition. In this case the rule may be enunciated as follows:

Napier's rule modified.

The cosine of any part in a Right-Angled Spherical Triangle is equal to the product of the cotangents of the adjacent parts or of the sines of the opposite parts.

The quantity sought being, by the last rule, determined from its

cosine there is no ambiguity whether it belong to the first or the second quadrant.

The chief advantage of Napier's rule is the great help it gives to memory. But in employing it, some inconvenience arises in so far as a required part may not always be determined independently of the other unknown parts, which then must be first determined. By employing independently of Napier's rule the ten equations previously established we can easily determine, in every case, any one of the three unknown parts independently of the two others.

Then the solution of a Right-Angled Spherical Triangle presents the six following cases:

- I. Then the hypotenuse and one oblique angle are given.
- II. When the hypotenuse and one side are given.
- III. When one oblique angle and its opposite side are given.
- IV. When one oblique angle and its adjacent side are given.
- V. When the two sides are given.
- VI. When the two oblique angles are given.

We shall proceed to show how each of these cases may be solved by Napier's rule, or Napier's rule modified, and by independent formulae.

Let the student be very particular to take the complement of the hypotenuse and of the two oblique angles, instead of these parts themselves, when employing Napier's rule; or the complement of the two sides including the right-angle, when using Napier's rule modified.

An accent annexed to the symbol of any part denotes, as previously stated, the complement of that part.



Case 1.

Given the hypotenuse a , and the angle B , to find b , c , C .

1. *Solution by Napier's rule.*

$$\sin b = \cos a' \cos B' \quad (51)$$

$$\sin c = \tan b \tan B' \quad (52)$$

$$\sin C' = \tan a' \tan b \quad (53)$$

2. *Solution by Napier's rule modified.*

$$\cos b' = \sin a \sin B \quad (54)$$

$$\cos c' = \cot b' \cot B \quad (55)$$

$$\cos C = \cot a \cot b' \quad (56)$$

3. *Solution by independent formulæ.*

By (35) $\sin b = \sin a \sin B \quad (57)$

By (43), $\tan c = \tan a \cos B \quad (58)$

By (40), $\cot C = \cos a \tan B \quad (59)$

Case 2.

Given the hypotenuse a and one side c .

1. *Solution by Napier's rule.*

$$\sin B' = \tan a' \tan c \quad (60)$$

$$\sin C' = \cos c \cos B' \quad (61)$$

$$\sin b = \cos a' \cos C' \quad (62)$$

2. *Solution by Napier's rule modified.*

$$\cos B = \cot a \cot c' \quad (63)$$

$$\cos C = \sin c' \sin B \quad (64)$$

$$\cos b' = \sin a \sin C \quad (65)$$

3. *Solution by independent formulæ.*

$$\text{By (43)} \quad \cos B = \frac{\tan c}{\tan a} \quad (66)$$

$$\text{By (36)} \quad \sin C = \frac{\sin c}{\sin a} \quad (67)$$

$$\text{By (39)} \quad \cos b = \frac{\cos a}{\cos c} \quad (68)$$

Case 3.

Given one oblique angle B and its opposite side b .

1. *Solution by Napier's rule*

$$\sin c = \tan b \tan B \quad (69)$$

$$\sin a' = \cos b \cos c \quad (70)$$

$$\sin C' = \tan a' \tan b \quad (71)$$

2. *Solution by Napier's rule modified.*

$$\cos c' = \cot b' \cot B \quad (72)$$

$$\cos a = \sin b' \sin c' \quad (73)$$

$$\cos C = \cot a \cot b' \quad (74)$$

3. *Solution by independent formulae.*

By (38) $\sin c = \frac{\tan b}{\tan B} \quad (75)$

By (35) $\sin a = \frac{\sin b}{\sin B} \quad (76)$

By (41) $\sin C = \frac{\cos B}{\cos b} \quad (77)$

Since two Right-Angled Spherical Triangles may be found on the surface of the sphere, having a side and its opposite angle the same in both, but of which the remaining sides and the remaining angle of the one are the supplements of the remaining sides and the remaining angle of the other, it follows that when, as in the preceding case, the given parts are one side and its opposite angle, the problem admits of two solutions, and there are two values for each sine.

It must not be inferred that in every case all the required parts are less than 90° in one triangle, and greater, therefore, in the other. The proper values for each triangle are selected by means of the principles given on the page 187.

In using the Napier's rule modified, only one value for each part is directly found, since it is determined by the cosine.

Case 4.

Given one oblique angle B . and its adjacent side c .

1. *Solution by Napier's rule.*

$$\sin C' = \cos c \cos B' \quad (78)$$

$$\sin a' = \tan B' \tan C' \quad (79)$$

$$\sin b = \cos a' \cos B' \quad (80)$$

2. *Solution by Napier's rule modified,*

$$\cos C = \sin c' \sin B \quad (81)$$

$$\cos a = \cot B \cot C \quad (82)$$

$$\cos b = \sin a \sin B \quad (83)$$

3. *Solution by independent formulæ.*

$$\text{By (42)} \quad \cos C = \cos c \sin B \quad (84)$$

$$\text{By (43)} \quad \cot a = \cot c \cos B \quad (85)$$

$$\text{By (38)} \quad \tan b = \sin c \tan B \quad (86)$$

Case 5.

Given the two sides b, c .

1. *Solution by Napier's rule.*

$$\sin a' = \cos b \cos c \quad (87)$$

$$\sin B' = \tan a' \tan c \quad (88)$$

$$\sin C' = \tan a' \tan b \quad (89)$$

2. *Solution by Napier's rule modified.*

$$\cos a = \sin b' \sin c' \quad (90)$$

$$\cos B = \cot a \cot c' \quad (91)$$

$$\cos C = \cot a \cot b' \quad (92)$$

3. *Solution by independent formulæ.*

$$\text{By (39)} \quad \cos a = \cos b \cos c \quad (93)$$

$$\text{By (38)} \quad \cot B = \cot b \sin c \quad (94)$$

$$\text{By (37)} \quad \cot C = \cot c \sin b \quad (95)$$

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Case 6.

Given the two oblique angles B, C

1. *Solution by Napier's rule.*

$$\sin a' = \tan B' \tan C' \quad (96)$$

$$\sin b = \cos a' \cos B' \quad (97)$$

$$\sin c = \cos a' \cos C' \quad (98)$$

2. *Solution by Napier's rule modified.*

$$\cos a = \cot B \cot C \quad (99)$$

$$\cos b = \sin a \sin B \quad (100)$$

$$\cos c = \sin a \sin C \quad (101)$$

3. *Solution by independent formulæ.*

By (40) $\cos a = \cot B \cot C \quad (102)$

By (41) $\cos b = \frac{\cos B}{\sin C} \quad (103)$

By (42) $\cos c = \frac{\cos C}{\sin B} \quad (104)$

Exercises.

N. B. 1. In using Napier's rule in either of its two forms, the complement of parts which exceed 90° will be negative, and the values of their sines, cosines, tangents and cotangents will be easily determined by means of the relations established on the page 42.

2. In using the logarithmic functions, the student must be cautioned to be particular to add or subtract 10 according to the rules given on the page 118:

1. Given $a = 91^\circ 42'$ *Answer* $b = 95^\circ 22' 30''$

$B = 95^\circ 6'$ $c = 71^\circ 32' 14''$

$C = 71^\circ 36' 47''$

2. Given $a = 140^\circ$ *Answer* $B = 32^\circ 8' 48''$

$c = 20^\circ$ $C = 115^\circ 42' 24''$

$C = 144^\circ 36' 28''$

- | | | |
|----|--|--|
| 3. | Given $B = 36^\circ$
$b = 26^\circ 4'$ | Answer $c = 42^\circ 19' 17''$
$a = 48^\circ 22' 52''$
$C = 64^\circ 14' 26''$
or $c = 137^\circ 40' 43''$
$a = 131^\circ 37' 8''$
$C = 115^\circ 45' 34''$ |
| 4. | Given $B = 44^\circ 50'$
$c = 54^\circ 30'$ | Answer $C = 65^\circ 49' 53''$
$a = 63^\circ 10' 4''$
$b = 38^\circ 59' 11''$ |
| 5. | Given $b = 116^\circ$
$c = 16'$ | Answer $a = 114^\circ 55' 20''$
$B = 97^\circ 39' 24''$
$C = 17^\circ 41' 40''$ |
| 6. | Given $B = 69^\circ 20'$
$c = 58^\circ 16'$ | Answer $a = 76^\circ 30' 37''$
$b = 65^\circ 28' 58''$
$c = 55^\circ 47' 46''$ |

Quadrantal Triangle.

A Quadrantal Triangle is a spherical Triangle having one of its side a quadrant. The polar triangle of the Right-Angled Triangle is therefore always a quadrantal triangle. Such triangles are generally avoided in practice. When unavoidable, they may be readily solved by means of the polar triangle.

Oblique Angled Spherical Triangle.

The solution of an Oblique -Angled Spherical Triangle present the six following cases:

- I. When two sides and the angle opposite to one of them are given.
- II. When two angles and the side opposite to one of them are given.

- III. When two sides and the included angle are given.
- IV. When two angles and the included side are given.
- V. When the three sides are given.
- VI. When the three angles are given.

Case 1.

Given two sides, a , b , and the angle A opposite one of them



By (18) $\sin B = \sin A \frac{\sin b}{\sin a}$ (105)

Then by Napier's Analog, (31)

$$\tan \frac{1}{2}c = \frac{\sin \frac{1}{2}A + B}{\sin \frac{1}{2}A - B} \tan \frac{1}{2}(a-b) \quad (106)$$

c could be determined also by any one of the three other Analogies.

$$\sin C = \sin A \frac{\sin c}{\sin a} \quad (107)$$

There are generally, in the preceding case, two solutions each of which will satisfy the conditions, when the required part is determined by its sine.

The ambiguity may be removed by the following principles:

1. The greater side must be opposite the greater angle, and conversely.
2. When the sum of the given sides is less than 180° , the angle opposite to the less side is acute.
3. When the sum of the two given sides exceeds 180° , the angle opposite to the greater side is obtuse.
4. When the sum of those sides is 180° , the sum of the opposite angles is the same.
5. When the given side not opposite to the given angle differs from 90° more than the given side opposite to it, the angle opposite to the first side must be in the same quadrant as that side.

When these principles, which are based on Spherical Geometry, fail in determining the species of the angle, it remains doubtful.

When C only is required to be known, it can be determined from (45) and (47), which give

$$\tan v = \tan b \cos A \quad (108)$$

$$\cos (c - v) = \frac{\cos a}{\cos b} \cos V \quad (109)$$

c will be the sum or difference of v and $(c - v)$ according as A and B should be of the same or of different species. If B is doubtful, c will be also doubtful, and, therefore either the sum or difference may be taken.

When C only is required to be known, it may be determined from (46) and (50) which give

$$\tan V = \frac{\cot A}{\cos b} \quad (110)$$

$$\cos (C - V) = \frac{\tan b}{\tan a} \cos V \quad (111)$$

C is the sum or the difference of V and $(C - V)$ according as A and B are of the same species or not. If B is doubtful, C is also doubtful, and either of the two values may be taken.

Case 2.

Given the two angles A , B , and the side a opposite to one of them

$$\text{By (17) } \sin b = \sin a \frac{\sin B}{\sin A} \quad (112)$$

Then c and C are found as in (104) and (105)

This case also admits of two solutions, except when the ambiguity is removed by the following principles:

1. When the sum of the given angles is less than 180° , the side opposite to the less is less than 90° .
2. If the sum of those angles exceeds 180° , the side opposite to the greater is greater than 90° .
3. If the sum of those angles is 180° , the sum of the opposite sides is the same.

If c or C only be required to be determined, operate with the formulae 45 (the first) and (48) for the former, and with (48) and (50) for the latter, in the same manner as before. The arcs v and $(c - v)$ are of the same species or not according as A and B must be; c will be equal to the sum or difference of v and $(c - v)$ according as A and B should be of the same species; V and $(C - V)$ are of the same species or not according as a and b should be; and the sum or difference will be taken for the value of c according as A and B are of the same or of different species.

Case 3.

Given two sides a, b , and the included angle C .

By Napier's Analogies (31) and (32) $\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ are determined. The sum of these results is the angle opposite the greater of the given angles; their difference is the other angle. Then $\frac{1}{2}c$ may be determined by either of the same Analogies.

If c only be required to be determined, proceed with (45) and (47) as before.

Case 4.

Given two angles A, B , and the included side c .

By Napier's Analogies (33) and (34) $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$ are determined, whence a and b are easily known. Then $\frac{1}{2}C$ is found by either (33) or (34).

If C only be required to be determined, proceed with (46) and (50) as before.

Case 5.

Given the three sides, a, b, c .

Any one of the angles may be found by one of the formulæ (2), (4), (5), (6), (7).

Case 6.

Given the three Angles A, B, C .

Any one of the three angles may be found by one of the formulæ (8), (10), (11), (12), (13).

Exercises.

1. Given $a = 124^\circ 53'$ *Answer* $B = 10^\circ 19' 34''$
 $b = 31^\circ 19'$ $C = 171^\circ 48' 22''$
 $A = 16^\circ 26'$ $c = 155^\circ 35' 22''$
2. Given $a = 115^\circ 28'$ *Answer* $B = 59^\circ 39'$
 $b = 60^\circ 29'$ $C = 172^\circ 43'$
 $A = 56^\circ 17'$ $c = 172^\circ 23'$
 or $B = 120^\circ 21'$
 $C = 72^\circ 52'$
 $c = 88^\circ 53'$
3. Given $A = 115^\circ 36' 45''$ *Answer* $a = 114^\circ 26' 50''$
 $B = 80^\circ 19' 12''$ $c = 82^\circ 33' 31''$
 $b = 84^\circ 21' 56''$ $C = 79^\circ 10' 30''$
4. Given $A = 61^\circ 37' 53''$ *Answer* $a = 42^\circ 37' 18''$
 $B = 139^\circ 54' 34''$ $c = 129^\circ 41' 5''$
 $b = 150^\circ 17' 26''$ $C = 89^\circ 54' 19''$
 or $a = 137^\circ 22' 42''$
 $c = 19^\circ 58' 35''$
 $C = 20^\circ 21' 18''$
5. Given $a = 78^\circ 15'$ *Answer* $A = 62^\circ 31' 40''$
 $b = 56^\circ 20'$ $B = 48^\circ 57' 29''$
 $C = 120^\circ$ $c = 107^\circ 7' 25''$
6. Given $A = 104^\circ 5'$ *Answer* $a = 102^\circ 17'$
 $B = 82^\circ 18'$ $b = 86^\circ 41'$
 $c = 71^\circ 45'$ $C = 70^\circ 31'$
7. Given $a = 71^\circ 15'$ *Answer* $A = 120^\circ 35' 55''$
 $b = 39^\circ 10'$ $B = 30^\circ 25' 34''$
 $c = 40^\circ 35'$ $C = 31^\circ 26' 32''$
8. Given $A = 109^\circ 55'$ *Answer* $a = 98^\circ 21' 20''$
 $B = 116^\circ 38'$ $b = 109^\circ 50' 10''$
 $C = 120^\circ 43'$ $c = 115^\circ 13' 7''$

END.

